Chapter 1

Derivatives and Limits

Differentiation is one of the two fundamental operations of calculus.

Differential calculus describes and analyzes change. The position of a moving object, the population of a city or a bacterial colony, the height of the sun in the sky, and the price of cheese all change with time. Altitude can change with position along a road; the pressure inside a balloon changes with temperature. To measure the rate of change in all these situations, we introduce in this chapter the operation of differentiation.

1.1 Introduction to the Derivative

Velocities and slopes are both derivatives.

This section introduces the basic idea of the derivative by studying two problems. The first is the problem of finding the velocity of a moving object, and the second is the problem of finding the slope of the line tangent to a graph.

To analyze velocity, imagine a bus which moves due east on a straight highway. Let x designate the time in seconds that has passed since we first observed the bus. (Using "x" for time rather than the more common "t" will make it easier to compare velocities with slopes.) Suppose that after x seconds the bus has gone a distance y meters to the east (Fig. 1.1.1). Since the distance y depends on the time x, we have a distance function $y = f(x)$. For example, if

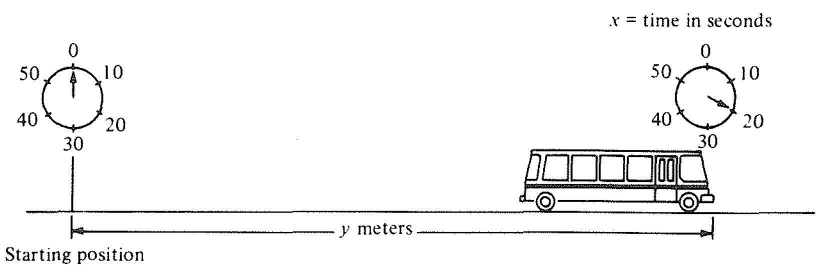


Figure 1.1.1. What is the velocity of the bus in terms of its position?

$f(x)$ happens to be $f(x)=2x^2$ for $0 \leqslant x \leqslant 5$, then the bus has gone $2 \cdot (3)^2 = 18$ meters after 3 seconds and $2 \cdot (5)^2 = 50$ meters after 5 seconds.

The velocity of the bus at any given moment, measured in meters per second, is a definite physical quantity; it can be measured by a speedometer on the bus or by a stationary radar device. Since this velocity refers to a single instant, it is called the *instantaneous velocity*. Given a distance function such as $y=f(x)=2x^2$, how can we calculate the instantaneous velocity at a specific time x_0, such as $x_0=3$ seconds? To answer this question, we will relate the instantaneous velocity to the average velocity during short time intervals.

Suppose that the distance covered is measured at time x_0 and again at a later time x; these distances are $y_0=f(x_0)$ and $y=f(x)$. Let $\Delta x = x - x_0$ designate the time elapsed between our two measurements.[1] Then the extra distance covered is $y - y_0$, which we designate by $\Delta y = y - y_0$. The *average velocity* during the time interval Δx is defined simply as the distance travelled divided by the elapsed time; that is, average velocity $= \Delta y/\Delta x = [f(x)-f(x_0)]/\Delta x$. Since $x = x_0 + \Delta x$, we can also write

$$\text{average velocity} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Example 1 A bus travels $2x^2$ meters in x seconds. Find Δx, Δy and the average velocity during the time interval Δx for the following situations: (a) $x_0=3$, $x=4$; (b) $x_0=3$, $x=3.1$; (c) $x_0=3$, $x=3.01$.

Solution (a) $\Delta x = x - x_0 = 4-3 = 1$ second, $\Delta y = f(x_0+\Delta x) - f(x_0) = f(4) - f(3) = 2\cdot 4^2 - 2\cdot 3^2 = 14$ meters, average velocity $= \Delta y/\Delta x = 14$ meters per second; (b) $\Delta x = 0.1$, $\Delta y = 1.22$, average velocity $= 12.2$; (c) $\Delta x = 0.01$, $\Delta y = .1202$, average velocity $= 12.02$ meters per second. ▲

If we specify the accuracy to which we want to determine the instantaneous velocity, we can expect to get this accuracy by calculating the average velocity $\Delta y/\Delta x$ for Δx sufficiently small. As the desired accuracy increases, Δx may need to be made even smaller; the *exact* velocity may then be described as the number v which $\Delta y/\Delta x$ approximates as Δx becomes very small. For instance, in Example 1, you might guess that the instantaneous velocity at $x_0=3$ seconds is $v=12$ meters per second; this guess is correct, as we will see shortly.

Our description of v as the number which $\Delta y/\Delta x$ approximates for Δx very small is a bit vague, because of the ambiguity in what is meant by "approximates" and "very small." Indeed, these ideas were the subject of controversy during the early development of calculus around 1700. It was thought that Δx ultimately becomes "infinitesimal," and for centuries people argued about what, if anything, "infinitesimal" might mean. Using the notion of "limit," a topic taken up in the next section, one can resolve these difficulties. However, if we work on an intuitive basis with such notions as "approximates," "gets close to," "small," "very small," "nearly zero," etc., we can solve problems and arrive at answers that will be fully justified later.

Example 2 The bus has gone $f(x)=2x^2$ meters at time x (in seconds). Calculate its instantaneous velocity at $x_0=3$.

[1] Δ is the capital Greek letter "delta," which corresponds to the Roman D and stands for difference. The combination "Δx", read "delta-x", is not the product of Δ and x but rather a single quantity: the difference between two values of x.

Solution We choose Δx arbitrarily and calculate the average velocity for a time interval Δx starting at time $x_0 = 3$:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(3+\Delta x)-f(3)}{\Delta x} = \frac{2(3+\Delta x)^2 - 2\cdot 3^2}{\Delta x}\\ &= \frac{2\left(9+6\Delta x+(\Delta x)^2\right)-2\cdot 9}{\Delta x} = \frac{18+12\Delta x+2(\Delta x)^2-18}{\Delta x}\\ &= \frac{12\Delta x+2(\Delta x)^2}{\Delta x} = 12+2\Delta x.\end{aligned}$$

If we let Δx become very small in this last expression, $2\Delta x$ becomes small as well, and so $\Delta y/\Delta x = 12 + 2\Delta x$ approximates 12. Thus the required instantaneous velocity at $x_0 = 3$ is 12 meters per second. Note how nicely the 18's cancelled. This allowed us to divide through by Δx and avoid ending up with a zero in the denominator. ▲

Warning In calculating what $\Delta y/\Delta x$ approximates for Δx nearly zero, it usually does no good to set $\Delta x = 0$ directly, for then we merely get $0/0$, which gives us no information.

The following more general procedure is suggested by Example 2.

Instantaneous Velocity

To calculate the instantaneous velocity at x_0 when the position at time x is $y = f(x)$:

1. Form the average velocity over the interval from x_0 to $x_0 + \Delta x$:
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$
2. Simplify your expression for $\Delta y/\Delta x$ as much as possible, cancelling Δx from numerator and denominator wherever you can.
3. Find the number v that is approximated by $\Delta y/\Delta x$ for Δx small.

Example 3 The position of a bus at time x is $y = 3x^2 + 8x$ for $x \geqslant 0$. (a) Find the instantaneous velocity at an arbitrary positive time x_0. (b) At what time is the instantaneous velocity 11 meters per second?

Solution (a) The calculation is similar to that of Example 2, except that x_0 no longer has the specific value $x_0 = 3$. The average velocity for a time interval Δx starting at x_0 is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

where $f(x) = 3x^2 + 8x$. Thus

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{\left[3(x_0+\Delta x)^2+8(x_0+\Delta x)\right]-(3x_0^2+8x_0)}{\Delta x}\\ &= \frac{6x_0\Delta x+3(\Delta x)^2+8\Delta x}{\Delta x} = 6x_0+8+3\Delta x.\end{aligned}$$

As Δx gets close zero, the term $3\Delta x$ gets close to zero as well, so $\Delta y/\Delta x$ gets close to (that is, approximates) $6x_0 + 8$. Thus our instantaneous velocity is $v = 6x_0 + 8$ meters per second at the positive time x_0.

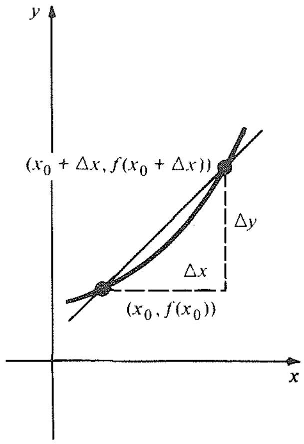


Figure 1.1.2. $\Delta y/\Delta x$ is the slope of the secant line.

(b) We set the velocity equal to 11: $6x_0 + 8 = 11$. Solving for x_0 gives $x_0 = \frac{1}{2}$ second. ▲

The second problem we study is a geometric one—to find the slope of the line tangent to the graph of a given function. We shall see that this problem is closely related to the problem of finding instantaneous velocities.

To solve the slope problem for the function $y = f(x)$, we begin by drawing the straight line which passes through the points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$, where Δx is a positive number; see Fig. 1.1.2. This straight line is called a *secant line*, and $\Delta y/\Delta x = [f(x_0 + \Delta x) - f(x_0)]/\Delta x$ is its slope.

As Δx becomes small, x_0 being fixed, it appears that the secant line comes close to the tangent line, so that the slope $\Delta y/\Delta x$ of the secant line comes close to the slope of the tangent line. See Fig. 1.1.3.

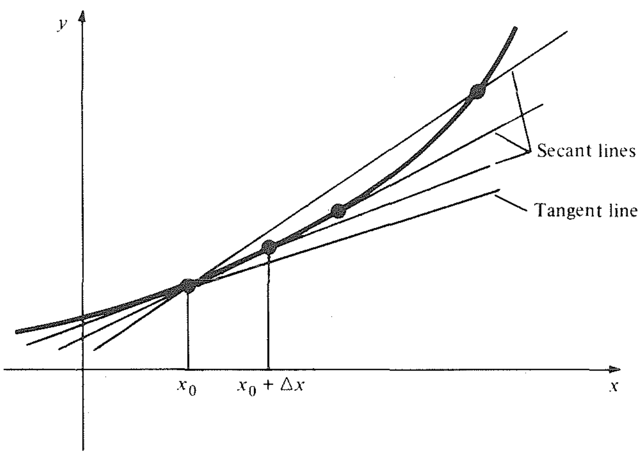


Figure 1.1.3. The secant line comes close to the tangent line as the second point moves close to x_0.

Slope of the Tangent Line

Given a function $y = f(x)$, the slope m of the line tangent to its graph at (x_0, y_0) is calculated as follows:

1. Form the slope of the secant line:
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$
2. Simplify the expression for $\Delta y/\Delta x$, cancelling Δx if possible.
3. Find the number m that is approximated by $\Delta y/\Delta x$ for Δx small.

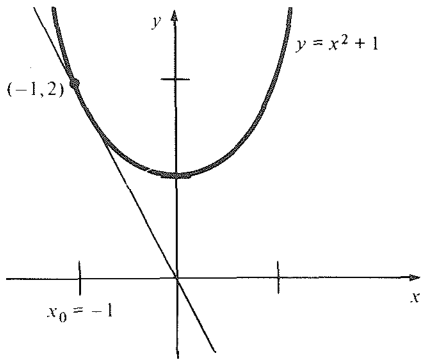
Example 4 Calculate the slope of the tangent line to the graph of $f(x) = x^2 + 1$ at $x_0 = -1$. Indicate your result on a sketch.

Solution We form the slope of the secant line:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\\ &= \frac{\left[(-1 + \Delta x)^2 + 1\right] - \left[(-1)^2 + 1\right]}{\Delta x} = \frac{-2\Delta x + (\Delta x)^2}{\Delta x}\\ &= -2 + \Delta x.\end{aligned}$$

For Δx small, this approximates -2, so the required slope is -2. Figure 1.1.4

Figure 1.1.4. The tangent line to $y = x^2 + 1$ at $x_0 = -1$ has slope -2.



shows the graph of the parabola $y = x^2 + 1$. We have sketched the tangent line through the point $(-1, 2)$. ▲

We define the *slope* of the graph of the function f at $(x_0, f(x_0))$ to be the slope of the tangent line there.

Up to this point, we have drawn all the pictures with Δx positive. However, the manipulations in Examples 2, 3, and 4 are valid if Δx has any sign, as long as $\Delta x \neq 0$. From now on we will allow Δx to be either positive or negative.

Comparing the two previous boxes, we see that the procedures for calculating instantaneous velocities and for calculating slopes are actually identical; for example, the velocity calculation of Example 2 also tells us the slope m of $y = 2x^2$ at $(3, 18)$, namely $m = 12$. We will later find that the same procedure applies to many other situations. It is thus convenient and economical to introduce terms which apply to all the different situations: instead of calling $\Delta y/\Delta x$ an average velocity or the slope of a secant, we call it a *difference quotient*; we call the final number obtained a *derivative* rather than an instantaneous velocity or a slope. We use the notation $f'(x_0)$ to designate the derivative of f at x_0.

The Derivative

To calculate the derivative $f'(x_0)$ of a function $y = f(x)$ at x_0:

1. Form the difference quotient
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$
2. Simplify $\Delta y/\Delta x$, cancelling Δx if possible.
3. The derivative is the number $f'(x_0)$ that $\Delta y/\Delta x$ approximates for Δx small.

This operation of finding a derivative is called *differentiation*.

The reader should be aware that the precise version of Step 3 involves the notion of a limit, which is discussed in the next section.

Example 5 Suppose that m is a constant. Differentiate $f(x) = mx + 2$ at $x_0 = 10$.

Solution Here the function is linear, so the derivative should be equal to the slope: $f'(10) = m$. To see this algebraically, calculate

$$\frac{\Delta y}{\Delta x} = \frac{[m(10 + \Delta x) + 2] - (m \cdot 10 + 2)}{\Delta x} = \frac{m\,\Delta x}{\Delta x} = m.$$

This approximates (in fact equals) m for Δx small, so $f'(10) = m$. ▲

Glancing back over our examples, we notice that all the functions have been either linear or quadratic. By treating a general quadratic function, we can check our previous results and point the way to the goal of developing general rules for finding derivatives.

Quadratic Function Rule

Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants, and let x_0 be any real number. Then $f'(x_0) = 2ax_0 + b$.

To justify the quadratic function rule, we form the difference quotient

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}\\ &= \frac{a(x_0+\Delta x)^2 + b(x_0+\Delta x) + c - ax_0^2 - bx_0 - c}{\Delta x}\\ &= \frac{ax_0^2 + 2ax_0\Delta x + a(\Delta x)^2 + bx_0 + b\,\Delta x + c - ax_0^2 - bx_0 - c}{\Delta x}\\ &= \frac{2ax_0\Delta x + a(\Delta x)^2 + b\,\Delta x}{\Delta x}\\ &= 2ax_0 + b + a\,\Delta x.\end{aligned}$$

As Δx approaches zero, $a\,\Delta x$ approaches zero, too, so $\Delta y/\Delta x$ approximates $2ax_0 + b$. Therefore $2ax_0 + b$ is the derivative of $ax^2 + bx + c$ at $x = x_0$.

Example 6 Find the derivative of $f(x) = 3x^2 + 8x$ at (a) $x_0 = -2$ and (b) $x_0 = \frac{1}{2}$.

Solution (a) Applying the quadratic function rule with $a = 3$, $b = 8$, $c = 0$, and $x_0 = -2$, we find $f'(-2) = 2(3)(-2) + 8 = -4$.

(b) Taking $a = 3$, $b = 8$, $c = 0$ and $x_0 = \frac{1}{2}$, we get $f'(\frac{1}{2}) = 2 \cdot 3 \cdot (\frac{1}{2}) + 8 = 11$, which agrees with our answer in Example 3(b). ▲

If we set $a = 0$ in the quadratic function rule, we find that the derivative of any linear function $bx + c$ is the constant b, independent of x_0: the slope of a linear function is constant. For a general quadratic function, though, the derivative $f'(x_0)$ does depend upon the point x_0 at which the derivative is taken. In fact, we can consider f' as a *new function*; writing the letter x instead of x_0, we have $f'(x) = 2ax + b$. We can rephrase the quadratic function rule with x_0 replaced by x as in the following box, which also summarizes the special cases $a = 0$ and $a = 0 = b$.

Differentiating the Simplest Functions

The derivative of the quadratic function $f(x) = ax^2 + bx + c$ is the linear function $f'(x) = 2ax + b$.

The derivative of the linear function $f(x) = bx + c$ is the constant function $f'(x) = b$.

The derivative of the constant function $f(x) = c$ is the zero function $f'(x) = 0$.

The next example illustrates the use of thinking of the derivative as a function.

Example 7 There is one point on the graph of the parabola $y = f(x) = x^2 - 4x + 5$ where the slope is zero, so that the tangent line is horizontal (Fig. 1.1.5). Find that point using: (a) derivatives; and (b) algebra.

Solution (a) By the quadratic function rule with $a = 1$, $b = -4$, $c = 5$, the derivative function is $f'(x) = 2x - 4$. For zero slope we have $0 = f'(x) = 2x - 4$, i.e., $x = 2$. Then $y = 1$, so our point is $(2, 1)$. This point is called the *vertex* of the parabola.

(b) Completing the square gives $f(x) = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1$. Now $(x - 2)^2$ is zero for $x = 2$ and positive otherwise, so the parabola has its lowest point at $x = 2$. It is plausible from the figure, and true, that this low point is the point where the slope is zero. ▲

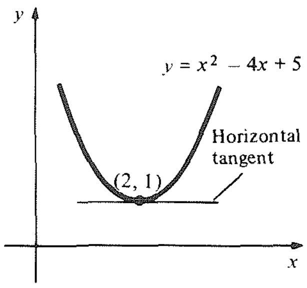


Figure 1.1.5. The vertex of the parabola is the point where its slope is zero.

We conclude this section with some examples of standard terms and notations. When we are dealing with functions given by specific formulas, we often omit the function names. Thus in Example 7(a) we can say "the derivative of $x^2 - 4x + 5$ is $2x - 4$." Another point is that we can use letters different from x, y, and f. For example, the area A of a circle depends on its radius r; we can write $A = g(r) = \pi r^2$. The quadratic function rule with $a = \pi$, $b = 0 = c$, with f replaced by g and with x replaced by r, tells us that $g'(r) = 2\pi r$. Thus for a circle the derivative of the area function is the circumference function—a fact whose geometric interpretation will be discussed in Section 2.1. Similarly, the time is often denoted by t in velocity problems.

Example 8 A stunt woman is on a moving passenger train. Her distance function is $3t^2 + t$. On the adjacent track is a long moving freight train. The distance function for the center of this freight train is $t^2 + 7t$. She must jump to the freight train. What time is best?

Solution The safest time to jump is when the stunt woman has the same velocity as the freight train (see Fig. 1.1.6). Her instantaneous velocity v is the derivative of

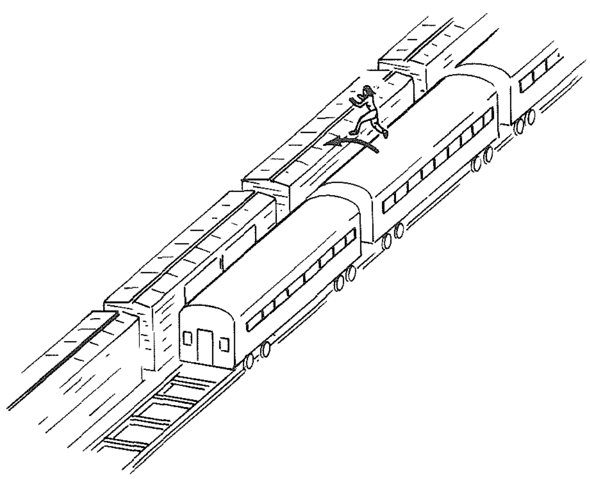


Figure 1.1.6. The stunt woman should jump when she has the same velocity as the freight train.

$3t^2 + t$. By the quadratic function rule, $v = 6t + 1$; similarly the instantaneous velocity of the freight train is $2t + 7$. The velocities are equal when $2t + 7 = 6t + 1$, i.e., $t = \frac{3}{2}$. That is the safest time. ▲

In this section, we have discussed the derivative, one of the two most basic concepts of calculus. We showed how to find derivatives in some cases and indicated a few of their applications. Before we can usefully discuss other applications of derivatives, we need to develop efficient techniques for calculating them. The next section begins that task.

Exercises for Section 1.1

In Exercises 1–4, y represents the distance a bus has travelled after x seconds. Find Δy and the average velocity during the time interval Δx for the following situations.

(a) $x_0 = 2$, $\Delta x = 0.5$ (b) $x_0 = 2$, $\Delta x = 0.01$
(c) $x_0 = 4$, $\Delta x = 0.1$ (d) $x_0 = 4$, $\Delta x = 0.01$

1. $y = x^2 + 3x$
2. $y = 3x^2 + x$
3. $y = x^2 + 10x$
4. $y = 2x$

In Exercises 5–8, $f(x)$ is the number of meters a bus has gone at a time x (in seconds). Find the instantaneous velocity at the given time x_0.

5. $x^2 + 3x$; $x_0 = 2$
6. $x^2 + 3x$; $x_0 = 4$
7. $3x^2 + x$; $x_0 = 2$
8. $3x^2 + x$; $x_0 = 4$

In Exercises 9–12, y is the position (measured in meters) of a bus at time x (in seconds). (a) Find the instantaneous velocity at an arbitrary (positive) time x_0. (b) At what time is the instantaneous velocity 10 meters per second?

9. $y = x^2 + 3x$
10. $y = 3x^2 + x$
11. $y = x^2 + 10x$
12. $y = 2x$

In Exercises 13–16, use the $\Delta y/\Delta x$ method of Example 4 to find the slope of the tangent line to the graph of the given function at the given point. Sketch.

13. $y = x^2$; $x_0 = 1$
14. $y = -x^2$; $x_0 = 2$
15. $y = 5x^2 - 3x + 1$; $x_0 = 0$
16. $y = x + 1 - x^2$; $x_0 = 2$

In Exercises 17–20 use the $\Delta y/\Delta x$ method of Example 5 to compute the derivative of $f(x)$ at x_0; a is a constant in each case.

17. $f(x) = ax + 2$; $x_0 = 0$
18. $f(x) = 2x + a$; $x_0 = 0$
19. $f(x) = ax^2$; $x_0 = 1$
20. $f(x) = 8x^2 + a$; $x_0 = 2$

In Exercises 21–24, use the quadratic function rule to find the derivative of the given function at the indicated point.

21. $f(x) = x^2 + x - 1$; $x_0 = 1$
22. $f(x) = x^2 - x$; $x_0 = 2$
23. $f(x) = 3x^2 + x - 2$; $x_0 = -2$
24. $f(x) = -3x^2 - x + 1$; $x_0 = -1$

In Exercises 25–28, find the vertex of the given parabola using (a) derivatives and (b) algebra.

25. $y = x^2 - 16x + 2$
26. $y = x^2 + 8x + 2$
27. $y = -2x^2 - 8x - 1$
28. $y = -2x^2 - 3x + 5$

Differentiate the functions in Exercises 29–36 using the quadratic function rule.

29. $f(x) = x^2 + 3x - 1$
30. $f(x) = -3x + 4$
31. $f(x) = (x - 1)(x + 1)$
32. $f(x) = (9 - x)(1 - x)$
33. $g(t) = -4t^2 + 3t + 6$
34. $g(r) = \pi r^2 + 3$
35. $g(s) = 1 - s^2$
36. $h(t) = 3t^2 - 5t + 9$
37. Inspector Clumseaux is on a moving passenger train. His distance function is $2t^2 + 3t$. On the adjacent track is a long moving freight train; the distance function for the center of the freight train is $3t^2 + t$. What is the best time for him to jump to the freight train?
38. Two trains, A and B, are moving on adjacent tracks with positions given by the functions $A(t) = t^2 + t + 5$ and $B(t) = 3t + 4$. What is the best time for a hobo on train B to make a moving transfer to train A?
39. An apple falls from a tall tree toward the earth. After t seconds, it has fallen $4.9t^2$ meters. What is the velocity of the apple when $t = 3$?
40. A rock thrown down from a bridge has fallen $4t + 4.9t^2$ meters after t seconds. Find its velocity at $t = 3$.
41. $f(x) = x^2 - 2$; find $f'(3)$
42. $f(x) = -13x^2 - 9x + 5$; find $f'(1)$
43. $f(x) = 1$; find $f'(7)$
44. $g(s) = 0$; find $g'(3)$
45. $k(y) = (y + 4)(y - 7)$; find $k'(-1)$
46. $x(f) = 1 - f^2$; find $x'(0)$
47. $f(x) = -x + 2$; find $f'(3.752764)$
48. $g(a) = 10a - 8$; find $g'(3.1415)$

In Exercises 49–54, find the derivative of each of the given functions by finding the value approximated by $\Delta y/\Delta x$ for Δx small:

49. $4x^2 + 3x + 2$
50. $(x - 3)(x + 1)$
51. $1 - x^2$
52. $-x^2$
53. $-2x^2 + 5x$
54. $1 - x$
55. Let $f(x) = 2x^2 + 3x + 1$. (a) For which values of x is $f'(x)$ negative, positive, and zero? (b) Identify these points on a graph of f.
56. Show that two quadratic functions which have the same derivative must differ by a constant.
57. Let $A(x)$ be the area of a square of side length x. Show that $A'(x)$ is half the perimeter of the square.
58. Let $A(r)$ be the area of a circle of radius r. Show that $A'(r)$ is the circumference.
59. Where does the line tangent to the graph of $y = x^2$ at $x_0 = 2$ intersect the x axis?
60. Where does the line tangent to the graph of $y = 2x^2 - 8x + 1$ at $x_0 = 1$ intersect the y axis?
61. Find the equation of the line tangent to the graph of $f(x) = 3x^2 + 4x + 2$ at the point where $x_0 = 1$. Sketch.
62. Find the tangent line to the parabola $y = x^2 - 3x + 1$ when $x_0 = 2$. Sketch.

★63. Find the lines through the point $(4, 7)$ which are tangent to the graph of $y = x^2$. Sketch. (*Hint*: Find and solve an equation for the x coordinate of the point of tangency.)

★64. Given a point $(\bar{x}, \bar{y})$, find a general rule for determining how many lines through the point are tangent to the parabola $y = x^2$.

★65. Let R be any point on the parabola $y = x^2$. Draw the horizontal line through R and draw the perpendicular to the tangent line at R. Show that the distance between the points where these lines cross the y axis is equal to $\frac{1}{2}$, regardless of the value of x. (Assume, however, that $x \neq 0$.)

★66. If $f(x) = ax^2 + bx + c = a(x - r)(x - s)$ (r and s are the roots of f), show that the values of $f'(x)$ at r and s are negatives of one another. Explain this by appeal to the symmetry of the graph.

★67. Using your knowledge of circles, sketch the graph of $f(x) = \sqrt{4 - x^2}$. Use this to guess the values of $f'(0)$ and $f'(\sqrt{2})$.

★68. A trained flea crawls along the parabola $y = x^2$ in such a way that its x coordinate at time t is $2t + 1$. The sun is shining from the east (positive x axis) so that a shadow of the flea is projected on a wall built along the y axis. What is the velocity of this shadow when $t = 3$?

★69. A ball is thrown upward at $t = 0$; its height in meters until it strikes the ground is $24.5t - 4.9t^2$ when the time is t seconds. Find:
(a) The velocity at $t = 0, 1, 2, 3, 4, 5$.
(b) The time when the ball is at its highest point.
(c) The time when the velocity is zero.
(d) The time when the ball strikes the ground.

★70. A toolbox falls from a building, its height y in feet from the ground after t seconds being given by $y = 100 - 16t^2$.
(a) Find the *impact time* t^*, i.e., the positive time for which $y = 0$.
(b) Find the *impact velocity*, i.e., the velocity at t^*.
(c) The *momentum* p is defined by $p = Wv/32$, where W is the weight in pounds, and v is the velocity in feet per second. Find the impact momentum for a 20-lb toolbox.

1.2 Limits

The limit of a function $f(t)$ at a point $x = x_0$ is the value which $f(x)$ approximates for x close to x_0.

In this section, we introduce limits and study their properties. In the following sections, we will use limits to clarify statements such as "$\Delta y/\Delta x$ approximates $f'(x_0)$ for Δx small," and to systematize the computing of derivatives. Some technical points in the theory of limits have been deferred to Chapter 11, where limits are needed again for other purposes. Readers who wish to see more of the theory now can read Section 11.1 together with the present section.

We illustrate the idea of a limit by looking at the function

$$f(x) = \frac{2x^2 - 7x + 3}{x - 3}$$

which is defined for all real numbers except 3. Computing values of $f(x)$ for some values of x near 3, we obtain the following tables:

x	3.5	3.1	3.01	3.0001	3.000001
$f(x)$	6	5.2	5.02	5.0002	5.000002

x	2.5	2.9	2.99	2.9999	2.999999
$f(x)$	4	4.8	4.98	4.9998	4.999998

It appears that, as x gets closer and closer to $x_0 = 3$, $f(x)$ gets closer and closer to 5, i.e., $f(x)$ approximates 5 for x close to 3. As in our discussion of the derivative, it does no good to set $x = 3$, because $f(3)$ is not defined. In the special case we are considering, there is another way to see that $f(x)$ approximates 5:

$$\frac{2x^2-7x+3}{x-3}=\frac{(2x-1)(x-3)}{(x-3)}=2x-1.$$

The cancellation of $(x-3)$ is valid for $x \neq 3$. Now for x close to 3, $2x-1$ approximates $2 \cdot 3 - 1 = 5$. Note that after cancelling $x-3$, the function becomes defined at $x_0 = 3$.

In general, suppose that we have a function $f(x)$ and are interested in its behavior near some value x_0. Assume that $f(x)$ is defined for all x near x_0, but not necessarily at $x = x_0$ itself. If the value $f(x)$ of f approximates a number l as x gets close to a number x_0, we say that "l is the limit of $f(x)$ as x

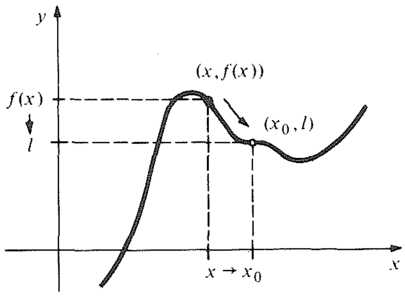


Figure 1.2.1. The notion of limit: as x approaches x_0, $f(x)$ gets near to l.

approaches x_0" or "$f(x)$ approaches l as x approaches x_0." See Fig. 1.2.1. Two usual notations for this are

$$f(x) \to l \quad \text{as} \quad x \to x_0$$

or

$$\lim_{x \to x_0} f(x) = l.$$

For example, the discussion above suggests that

$$\frac{2x^2-7x+3}{x-3} \to 5 \quad \text{as} \quad x \to 3;$$

that is

$$\lim_{x \to 3} \frac{2x^2-7x+3}{x-3} = 5.$$

Example 1 Using numerical computations, guess the value of $\lim_{x \to 4}[1/(4x-2)]$.

Solution We make a table using a calculator and round off to three significant figures:

x	4.1	4.01	4.001	3.9	3.99	3.999
$1/(4x-2)$	0.0694	0.0712	0.0714	0.0735	0.0716	0.0714

It appears that the limit is a number which, when rounded to three decimal places, is 0.071. In addition, we may notice that as $x \to 4$, the expression $4x-2$ in the denominator of our fraction approaches 14. The decimal expansion of $\frac{1}{14}$ is 0.071428 . . . , so we may guess that

$$\lim_{x \to 4} \frac{1}{4x-2} = \frac{1}{14}. \blacktriangle$$

We summarize the idea of limit in the following display.

> **The Notion of Limit**
>
> If the value of $f(x)$ approximates the number l for x close to x_0, then we say that f *approaches the limit* l *as* x *approaches* x_0, and we write
>
> $$f(x) \to l \quad \text{as} \quad x \to x_0, \quad \text{or} \quad \lim_{x \to x_0} f(x) = l.$$

The following points should be noted.

1. The quantity $\lim_{x \to x_0} f(x)$ depends upon the values of $f(x)$ for x near x_0, but not for x equal to x_0. Indeed, even if $f(x_0)$ is defined, it can be changed arbitrarily without affecting the value of the limit.
2. As x gets nearer and nearer to x_0, the values of $f(x)$ might not approach any fixed number. In this case, we say that $f(x)$ has no limit as $x \to x_0$, or that $\lim_{x \to x_0} f(x)$ does not exist.
3. In determining $\lim_{x \to x_0} f(x)$, we must consider values of x on both sides of x_0.
4. Just as in our discussion of the derivative, one can still legitimately complain that the definition of limit given in the preceding display is too vague. Readers who wish to see an air-tight definition should now read the first few pages of Section 11.1. (Section 11.1 is needed for other theoretical points in Chapter 11 and for proofs, but not for what follows here.)

Example 2 Reading the graph in Fig. 1.2.2, find $\lim_{t \to b} g(t)$ if it exists, for $b = 1, 2, 3, 4$, and 5.

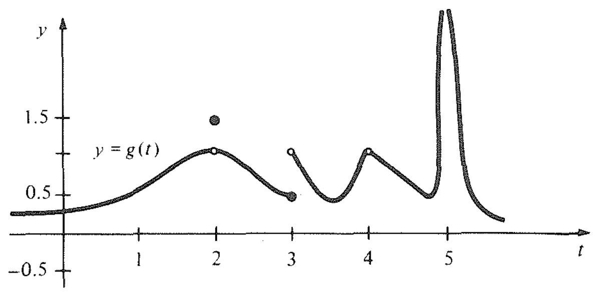


Figure 1.2.2. Find the limits of g at the indicated points. A small circle means that the indicated point does not belong to the graph.

Solution Notice first of all that we have introduced new letters; $\lim_{t \to b} g(t)$ means the value approached by $g(t)$ as t approaches b.

- $b = 1$: $\lim_{t \to 1} g(t) = 0.5$. In this case, $g(b)$ is defined and happens to be equal to the limit.
- $b = 2$: $\lim_{t \to 2} g(t) = 1$. In this case, $g(b)$ is defined and equals 1.5, which is not the same as the limit.
- $b = 3$: $\lim_{t \to 3} g(t)$ does not exist. For t near 3, $g(t)$ has values near 0.5 (for $t < 3$) and near 1 (for $t > 3$). There is no *single* number approached by $g(t)$ as t approaches 3.
- $b = 4$: $\lim_{t \to 4} g(t) = 1$. In this case, $g(b)$ is not defined.
- $b = 5$: $\lim_{t \to 5} g(t)$ does not exist. As t approaches 5, $g(t)$ grows larger and larger and does not approach any limit. ▲

The computation of limits is aided by certain properties, which we list in the following display. We will make no attempt to prove them until Chapter 11. Instead, we will present some remarks and graphs which suggest that they are reasonable.

Basic Properties of Limits

Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist:

Sum rule:

$$\lim_{x\to x_0}\left[f(x)+g(x)\right]=\lim_{x\to x_0} f(x)+\lim_{x\to x_0} g(x)$$

Product rule:

$$\lim_{x\to x_0}\left[f(x)g(x)\right]=\left[\lim_{x\to x_0} f(x)\right]\left[\lim_{x\to x_0} g(x)\right]$$

Reciprocal rule:

$$\lim_{x\to x_0}\left[\frac{1}{f(x)}\right]=\frac{1}{\lim_{x\to x_0} f(x)} \qquad \text{if} \quad \lim_{x\to x_0} f(x)\neq 0$$

Constant function rule:

$$\lim_{x\to x_0} c=c$$

Identity function rule:

$$\lim_{x\to x_0} x=x_0$$

Replacement rule: If the functions f and g have the same values for all x near x_0, but not necessarily including $x=x_0$, then

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x).$$

The sum and product rules are based on the following observation: If we replace the numbers y_1 and y_2 by numbers z_1 and z_2 which are close to y_1 and y_2, then z_1+z_2 and z_1z_2 will be close to y_1+y_2 and y_1y_2, respectively. Similarly, the reciprocal rule comes from such common sense statements as "$1/14.001$ is close to $1/14$."

The constant function rule says that if $f(x)$ is identically equal to c, then $f(x)$ is near c for all x near x_0. This is true because c is near c.

The identity function rule is true since it merely says that x is near x_0 if x is near x_0. Illustrations of the constant function rule and the identity function rule are presented in Fig. 1.2.3.

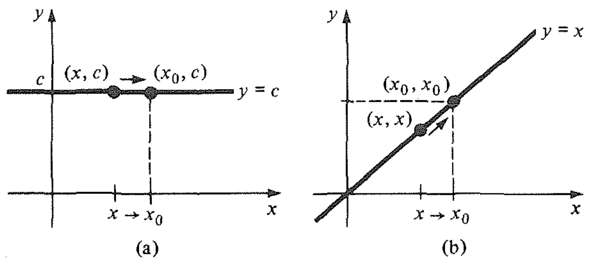


Figure 1.2.3. In (a) $\lim_{x\to x_0} c=c$ and in (b) $\lim_{x\to x_0} x=x_0$.

Finally, the replacement rule follows from the fact that $\lim_{x\to x_0} f(x)$ depends only on the values of $f(x)$ for x near x_0, and not at x_0 nor on values of x far away from x_0. The situation is illustrated in Fig. 1.2.4.

Example 3 Use the basic properties of limits: (a) to find $\lim_{x\to 3}(x^2+2x+5)$; (b) to show $\lim_{x\to 3}[(2x^2-7x+3)/(x-3)]=5$ as we guessed in the introductory calculation at the start of this section, and (c) to find $\lim_{u\to 2}[(8u^2+2)/(u-1)]$.

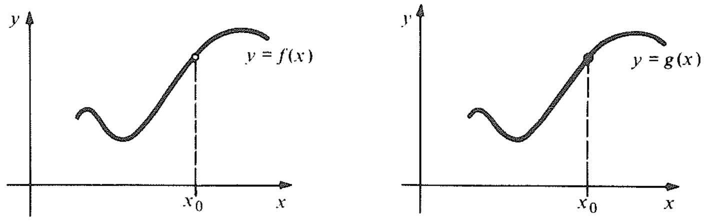


Figure 1.2.4. If the graphs of f and g are identical near x_0, except possibly at the single point where $x = x_0$, then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$.

Solution (a) Common sense suggests that the answer should be $3^2 + 2\cdot 3 + 5 = 20$. In fact this is correct.

By the product and identity function rules,

$$\lim_{x\to 3} x^2 = \lim_{x\to 3}(x\cdot x) = \Big(\lim_{x\to 3} x\Big)\Big(\lim_{x\to 3} x\Big) = 3\cdot 3 = 9.$$

By the product, constant function, and identity function rules,

$$\lim_{x\to 3} 2x = \Big(\lim_{x\to 3} 2\Big)\Big(\lim_{x\to 3} x\Big) = 2\cdot 3 = 6.$$

By the sum rule,

$$\lim_{x\to 3}(x^2 + 2x) = \lim_{x\to 3} x^2 + \lim_{x\to 3} 2x = 9 + 6 = 15.$$

Finally, by the sum and constant function rules,

$$\lim_{x\to 3}(x^2 + 2x + 5) = \lim_{x\to 3}(x^2 + 2x) + \lim_{x\to 3} 5 = 15 + 5 = 20.$$

(b) We cannot use common sense or the quotient rule, since

$$\lim_{x\to 3}(x-3) = \lim_{x\to 3} x - \lim_{x\to 3} 3 = 3 - 3 = 0.$$

Since substituting $x = 3$ into the numerator yields zero, $x - 3$ must be a factor; in fact, $2x^2 - 7x + 3 = (2x-1)(x-3)$, and we have

$$\frac{2x^2 - 7x + 3}{x-3} = \frac{(2x-1)(x-3)}{x-3}.$$

For $x \neq 3$, we can divide numerator and denominator by $x - 3$ to obtain $2x - 1$. Now we apply the replacement rule, with

$$f(x) = \frac{2x^2 - 7x + 3}{x-3} \quad \text{and} \quad g(x) = 2x - 1$$

since these two functions agree for $x \neq 3$. Therefore

$$\lim_{x\to 3} \frac{2x^2 - 7x + 3}{x-3} = \lim_{x\to 3}(2x-1) = 2\Big(\lim_{x\to 3} x\Big) - 1 = 2\cdot 3 - 1 = 5.$$

(c) Here the letter "u" is used in place of "x," but we do not need to change our procedures. By the sum, identity, and constant function rules, we get $\lim_{u\to 2}(u-1) = \lim_{u\to 2} u - \lim_{u\to 2} 1 = 2 - 1 = 1$. Similarly,

$$\begin{aligned}
&\lim_{u\to 2}(8u^2 + 2)\\
&\quad = \lim_{u\to 2} 8u^2 + \lim_{u\to 2} 2 && \text{(sum rule)}\\
&\quad = \Big(\lim_{u\to 2} 8\Big)\Big(\lim_{u\to 2} u^2\Big) + 2 && \text{(product and constant function rules)}\\
&\quad = 8\Big(\lim_{u\to 2} u\Big)\Big(\lim_{u\to 2} u\Big) + 2 && \text{(product and constant function rules)}\\
&\quad = 8\cdot 2\cdot 2 + 2 = 34 && \text{(identity function rule).}
\end{aligned}$$

Thus, by the product and reciprocal rules,

$$\lim_{u\to 2}\frac{8u^2+2}{u-1}=\lim_{u\to 2}\left[(8u^2+2)\cdot\frac{1}{(u-1)}\right]$$
$$=\lim_{u\to 2}(8u^2+2)\cdot\frac{1}{\lim_{u\to 2}(u-1)}=34\cdot\frac{1}{1}=34.$$

This agrees with the common sense rule obtained by substituting $u=2$. ▲

As you gain experience with limits, you can eliminate some of the steps used in the solution of Example 3. Moreover, you can use some further rules which can be derived from the basic properties.

Derived Properties of Limits

Assume that the limits on the right-hand sides below exist. Then we have:

Extended sum rule:

$$\lim_{x\to x_0}\left[f_1(x)+\cdots+f_n(x)\right]=\lim_{x\to x_0}f_1(x)+\cdots+\lim_{x\to x_0}f_n(x)$$

Extended product rule:

$$\lim_{x\to x_0}\left[f_1(x)\cdots f_n(x)\right]=\lim_{x\to x_0}f_1(x)\ldots\lim_{x\to x_0}f_n(x)$$

Constant multiple rule:

$$\lim_{x\to x_0}cf(x)=c\lim_{x\to x_0}f(x)$$

Quotient rule:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\lim_{x\to x_0}f(x)}{\lim_{x\to x_0}g(x)}\qquad\text{if}\quad\lim_{x\to x_0}g(x)\neq 0$$

Power rule:

$$\lim_{x\to x_0}x^n=x_0^n$$

$(n=0,\pm 1,\pm 2,\pm 3,\ldots$ and $x_0\neq 0$ if n is not positive).

We outline how these derived properties can be obtained from the basic properties. To prove the extended sum rule with three summands from the basic properties of limits, we must work out $\lim_{x\to x_0}(f_1(x)+f_2(x)+f_3(x))$ when $\lim_{x\to x_0}f_i(x)$ is known to exist. The idea is to use the basic sum rule for two summands. In fact $f_1(x)+f_2(x)+f_3(x)=f_1(x)+g(x)$, where $g(x)=f_2(x)+f_3(x)$. Note that $\lim_{x\to x_0}g(x)=\lim_{x\to x_0}f_2(x)+\lim_{x\to x_0}f_3(x)$ by the basic sum rule. Moreover $\lim_{x\to x_0}(f_1(x)+g(x))=\lim_{x\to x_0}f_1(x)+\lim_{x\to x_0}g(x)$ by the same rule. Putting these results together, we have

$$\lim_{x\to x_0}\left[f_1(x)+f_2(x)+f_3(x)\right]=\lim_{x\to x_0}\left[f_1(x)+g(x)\right]$$
$$=\lim_{x\to x_0}f_1(x)+\lim_{x\to x_0}g(x)$$
$$=\lim_{x\to x_0}f_1(x)+\lim_{x\to x_0}f_2(x)+\lim_{x\to x_0}f_3(x),$$

as we set out to show. The extended sum rule with more than three terms is now plausible; it can be proved by induction (see Exercise 65). The extended product rule can be proved by very similar arguments. To get the constant multiple rule, we may start with the basic product rule $\lim_{x\to x_0}[f(x)g(x)] = [\lim_{x\to x_0}f(x)][\lim_{x\to x_0}g(x)]$. Let $g(x)$ be the constant function $g(x)=c$; the constant function rule gives $\lim_{x\to x_0}[cf(x)] = [\lim_{x\to x_0}c][\lim_{x\to x_0}f(x)] = c\lim_{x\to x_0}f(x)$, as we wanted to show. Similarly, the quotient rule follows from the basic product rule and the reciprocal rule by writing $f/g = f\cdot 1/g$. The power rule follows from the extended product rule with $f_1(x)=x,\ldots,$ $f_n(x)=x$ and the identity function rule. The next example illustrates the use of the derived properties.

Example 4 Find $\displaystyle\lim_{x\to 1}\frac{x^3-3x^2+14x}{x^6+x^3+2}$.

Solution Common sense correctly suggests that the answer is $(1^3-3\cdot 1^2+14\cdot 1)/(1^6+1^3+2)=3$. To get this answer systematically, we shall write $f(x)=x^3-3x^2+14x$, $g(x)=x^6+x^3+2$, and use the quotient rule. First of all, $\lim_{x\to 1}x^6 = 1^6 = 1$ and $\lim_{x\to 1}x^3 = 1$ by the power rule; $\lim_{x\to 1}2 = 2$ by the constant function rule; since all three limits exist, $\lim_{x\to 1}g(x) = 1+1+2=4$ by the extended sum rule. Similarly, $\lim_{x\to 1}f(x)=12$. Since $\lim_{x\to 1}g(x)\neq 0$, the quotient rule applies and so $\lim_{x\to 1}[f(x)/g(x)] = \frac{12}{4} = 3$, as we anticipated. ▲

Clearly the common sense method of just setting $x=1$ is far simpler when it works. A general term to describe those situations where it does work is "continuity."

> **Definition of Continuity**
>
> A function $f(x)$ is said to be continuous at $x=x_0$ if $\lim_{x\to x_0}f(x)=f(x_0)$.

Thus if $f(x)$ is continuous at x_0, two things are true: (1) $\lim_{x\to x_0}f(x)$ exists and (2) this limit can be calculated by merely setting $x=x_0$ in $f(x)$, much as in Example 4. The geometric meaning of continuity will be analyzed extensively in Section 3.1.

We now discuss certain functions which are continuous at many or all values of x_0. Instead of the specific function $(x^3-3x^2+14x)/(x^6+x^3+2)$, we consider more generally a ratio $r(x)=f(x)/g(x)$ of two polynomials. Such a ratio is called a *rational function*, just as a ratio of integers is called a rational number. Note that a polynomial $f(x)$ is itself a rational function—we can simply choose the denominator $g(x)$ in the ratio $r(x)$ to be $g(x)=1$. Suppose that we are interested in the rational function $r(x)=f(x)/g(x)$ for values of x near x_0. Moreover, suppose that $g(x_0)\neq 0$ so that $r(x_0)$ is defined; for instance, in Example 4 we had $g(x_0)=4\neq 0$ at $x_0=1$. Using the limit rules in almost exactly the same way as we did in Example 4 leads to the conclusion that the common sense approach works for the rational function $r(x)$. We summarize in the following box.

> **Continuity of Rational Functions**
>
> If $f(x)$ is a polynomial or a ratio of polynomials and $f(x_0)$ is defined, then
>
> $$\lim_{x\to x_0}f(x)=f(x_0).$$

As an example of the use of the continuity of rational functions, note that to calculate $\lim_{x\to 4}[1/(4x-2)]$, we can now just set $x = 4$ to get $\frac{1}{14}$, as we guessed in Example 1 above. Indeed, students seduced by the simplicity of this rule often believe that a *limit* is nothing more than a *value*. The next example should help you avoid this trap.

Example 5 Find

$$\text{(a)}\quad \lim_{x\to 2}\frac{x^2+x-6}{x^2+2x-8}$$

and

$$\text{(b)}\quad \lim_{\Delta x\to 0}\frac{(\Delta x)^2+2\Delta x}{(\Delta x)^2+\Delta x},$$

where Δx is a variable.

Solution (a) The denominator vanishes when $x = 2$, so we cannot use the continuity of rational functions as yet. Instead we factor. When the denominator is not zero we have

$$\frac{x^2+x-6}{x^2+2x-8}=\frac{(x+3)(x-2)}{(x+4)(x-2)}=\frac{x+3}{x+4}.$$

Thus

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+x-6}{x^2+2x-8}&=\lim_{x\to 2}\frac{x+3}{x+4} && \text{(by the replacement rule)}\\ &=\frac{2+3}{2+4}=\frac{5}{6} && \text{(by the continuity of rational functions).}\end{aligned}$$

(b) The denominator vanishes when $\Delta x = 0$, so again we use the replacement rule:

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{(\Delta x)^2+2\Delta x}{(\Delta x)^2+\Delta x}&=\lim_{\Delta x\to 0}\frac{\Delta x+2}{\Delta x+1} && \text{(replacement rule)}\\ &=2 && \text{(continuity of rational functions).} \blacktriangle\end{aligned}$$

There are many limits that cannot be dealt with by the laws of limits we have so far. For example, we claim that if x_0 is positive, then

$$\lim_{x\to x_0}\sqrt{x}=\sqrt{x_0},$$

i.e., the function $f(x)=\sqrt{x}$ is continuous at x_0. To make this result plausible, *assume* that $\lim_{x\to x_0}\sqrt{x}=l$ exists. Then by the product rule,

$$l^2=\Big(\lim_{x\to x_0}\sqrt{x}\Big)\Big(\lim_{x\to x_0}\sqrt{x}\Big)=\lim_{x\to x_0}x=x_0.$$

Now l must be positive since $\sqrt{x}>0$ for all x which are positive, and all x which are close enough to x_0 are positive. Hence, $l=\sqrt{x_0}$. This limit is consistent with the appearance of the graph of $y=\sqrt{x}$. (See Fig. 1.2.5.)

In Section 11.1, we give a careful proof of the continuity of $\sqrt{x}$.

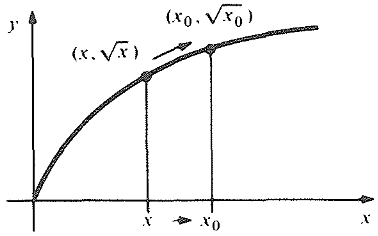


Figure 1.2.5. The graph of $y = \sqrt{x}$ suggests that $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

Example 6 Find

$$\lim_{x \to 3} \frac{8x^2}{1 + \sqrt{x}}.$$

Solution By using the properties of limits and the continuity of $\sqrt{x}$, we get $\lim_{x \to 3}(1 + \sqrt{x}) = \lim_{x \to 3} 1 + \lim_{x \to 3} \sqrt{x} = 1 + \sqrt{x} \neq 0$. Thus

$$\lim_{x \to 3} \frac{8x^2}{1 + \sqrt{x}} = \frac{\lim_{x \to 3} 8x^2}{\lim_{x \to 3}(1 + \sqrt{x})} = \frac{8 \cdot 3^2}{1 + \sqrt{3}} = \frac{72}{1 + \sqrt{3}}. \blacktriangle$$

Sometimes limits can fail to exist even when a function is given by a simple formula; the following is a case in point.

Example 7 Does $\lim_{x \to 0}(|x|/x)$ exist?

Solution The function in question has the value 1 for $x > 0$ and -1 for $x < 0$. For $x = 0$, it is undefined. (See Fig. 1.2.6.) There is no number l which is

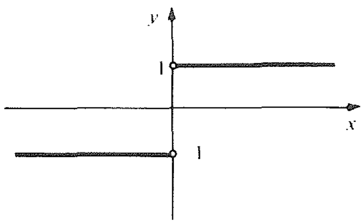


Figure 1.2.6. The graph of the function $|x|/x$.

approximated by $|x|/x$ as $x \to 0$, since $|x|/x$ is sometimes 1 and sometimes -1, according to the sign of x. We conclude that $\lim_{x \to 0}(|x|/x)$ does not exist. ▲

It is possible to define a notion of *one-sided limit* so that a function like $|x|/x$ has limits from the left and right (see Section 11.1 for details). Since the one-sided limits are different, the limit *per se* does not exist. The reader might wonder if any function of interest in applications actually shows a jump similar to that in Fig. 1.2.6. The answer is "yes." For example, suppose that a ball is dropped and, at $t = 0$, bounces off a hard floor. Its velocity will change very rapidly from negative (that is, downward) to positive (that is, upward). It is often convenient to idealize this situation by saying that the velocity function jumps from a negative to a positive value exactly at $t = 0$, much as in Fig. 1.2.6.

We conclude this section with some limits involving $\pm\infty$. We shall be quite informal and emphasize examples, again leaving a more careful discussion to Chapter 11. First, it is often useful to consider limits of the form $\lim_{x \to \infty} f(x)$. This symbol refers to the value approached by $f(x)$ as x becomes arbitrarily large. Likewise, $\lim_{x \to -\infty} f(x)$ is the value approached by $f(x)$ as x gets large in the negative sense. Limits as $x \to \pm\infty$ obey similar rules to those with $x \to x_0$.

Example 8 Find

(a) $\lim_{x\to\infty} \frac{1}{x}$;

(b) $\lim_{x\to\infty} \frac{2x+1}{3x+1}$;

and

(c) $\lim_{x\to-\infty} \frac{5x^2-3x+2}{x^2+1}$.

Solution As x gets very large, $1/x$ gets very small. Thus

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$.

We shall do (b) and (c) by writing the given expression in terms of $1/x$.

(b) $\lim_{x\to\infty} \frac{2x+1}{3x+1} = \lim_{x\to\infty} \frac{2+1/x}{3+1/x} = \frac{2+0}{3+0} = \frac{2}{3}$,

(c) $$\lim_{x\to-\infty} \frac{5x^2-3x+2}{x^2+1} = \lim_{x\to-\infty} \frac{5-3/x+2/x^2}{1+1/x^2} = \frac{5-0+0}{1+0} = 5. \blacktriangle$$

Example 9 Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the function f in Figure 1.2.7.

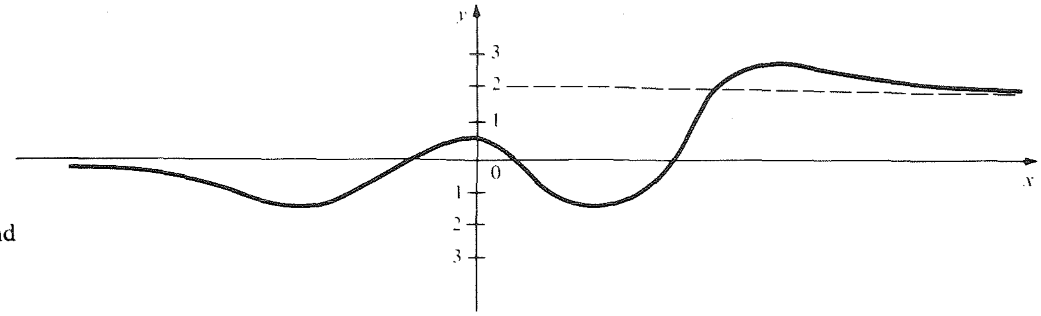


Figure 1.2.7. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Solution Assuming that the ends of the graph continue as they appear to be going, we conclude that $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 0$. ▲

Another kind of limit occurs when the value of $f(x)$ becomes arbitrarily large and positive as x approaches x_0. We then write $\lim_{x\to x_0} f(x) = \infty$. In this case $\lim_{x\to x_0} f(x)$ does not, strictly speaking, exist (infinity is not a real number). Similarly $\lim_{x\to x_0} f(x) = -\infty$ is read "the limit of $f(x)$ as x approaches x_0 is minus infinity," which means that while $\lim_{x\to x_0} f(x)$ does not exist, as x approaches x_0 from either side $f(x)$ becomes arbitrarily large in the negative sense.

Example 10 Find

(a) $\lim_{x\to2} \frac{-3x}{x^2-4x+4}$

and

(b) $\lim_{x\to0} \frac{3x+2}{x}$.

Solution (a) The denominator vanishes when $x = 2$, so the quotient rule does not apply. We may factor the denominator to get $-3x/(x^2-4x+4) = -3x/(x-2)^2$.

For x near 2, the numerator is near -6, while the denominator is small and positive, so the quotient is large and negative. Thus,

$$\lim_{x\to 2}\frac{-3x}{(x-2)^2} = -\infty.$$

(See Fig. 1.2.8(a).)

(b) We write $(3x+2)/x = 3 + 2/x$. When x is near 0, $2/x$ is *either* large and positive *or* large and negative, according to the sign of x (Fig. 1.2.8(b)). Hence $\lim_{x\to 0}[(3x+2)/x]$ does not have any value, finite or infinite. (To get $+\infty$ or $-\infty$, one-sided limits must be used.) ▲

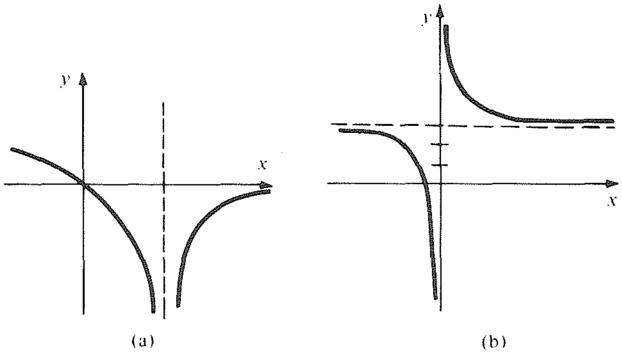


Figure 1.2.8. In (a) the limit is $-\infty$ and in (b) it does not exist.

$\lim_{x\to 2}\dfrac{-3x}{x^2-4x+4} = -\infty$ $y = \dfrac{3x+2}{x}$ is infinitely schizophrenic near $x=0$

Exercises for Section 1.2

1. Guess $\lim_{x\to 1}[(x^3-3x^2+5x-3)/(x-1)]$ by doing numerical calculations. Verify your guess by using the properties of limits.
2. Find $\lim_{x\to -1}[2x/(4x^2+5)]$, first by numerical calculation and guesswork, then by the basic properties of limits, and finally by the continuity of rational functions.

Refer to Fig. 1.2.9 for Exercises 3 and 4.

3. Find $\lim_{x\to -3} f(x)$ and $\lim_{x\to 3} f(x)$ if they exist.
4. Find $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ if they exist.

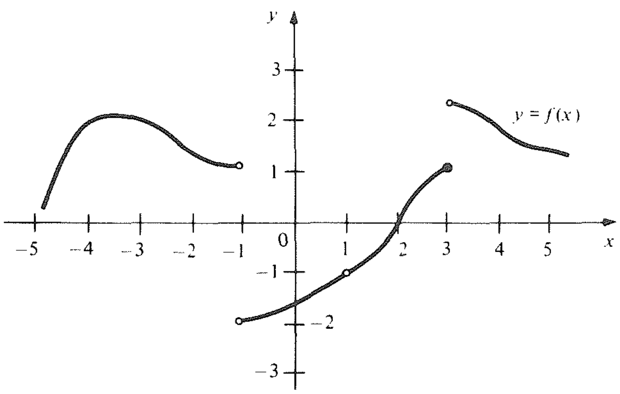


Figure 1.2.9. Find the limits at $x = -3, -1, 1$, and 3 if they exist. A small circle means that the indicated point does not belong to the graph.

Use the basic properties of limits to find the limits in Exercises 5–8.

5. $\lim_{x\to 3}(17+x)$
6. $\lim_{x\to 3} x^2$
7. $\lim_{u\to -1}\dfrac{u+1}{u-1}$
8. $\lim_{s\to 2}\dfrac{s^2-1}{s}$

Use the basic and derived properties of limits to find the limits in Exercises 9–12.

9. $\lim_{x\to 3}\dfrac{x^2-9}{x^2+3}$
10. $\lim_{x\to 2}\dfrac{(x^2+3x-10)}{(x+2)}$
11. $\lim_{x\to 1}\dfrac{x^{10}+8x^3-7x^2-2}{x+1}$
12. $\lim_{x\to 2}\dfrac{(x^2+3x-9)}{x+2}$

Use the continuity of rational functions and the replacement rule, if necessary, to evaluate the limits in Exercises 13–22.

13. $\lim_{u\to\sqrt{3}}\dfrac{u-\sqrt{3}}{u^2-3}$
14. $\lim_{t\to\sqrt{5}}\dfrac{t-\sqrt{5}}{t^2-5}$
15. $\lim_{x\to 2}\dfrac{x-2}{x-2}$
16. $\lim_{x\to 3}\dfrac{x^2-3}{x^2-3}$
17. $\lim_{x\to 3}\dfrac{x^2-4x+3}{x^2-2x-3}$
18. $\lim_{x\to -5}\dfrac{x^2+x-20}{x^2+6x+5}$
19. $\lim_{\Delta x\to 0}\dfrac{(\Delta x)^2+3(\Delta x)}{\Delta x}$
20. $\lim_{\Delta x\to 0}\dfrac{(\Delta x)^3+2(\Delta x)^2}{\Delta x}$

21. $\lim\limits_{\Delta x \to 0} \dfrac{3(\Delta x)^2 + 2(\Delta x)}{\Delta x}$

22. $\lim\limits_{\Delta x \to 0} \dfrac{(\Delta x)^3 + 2(\Delta x)^2 + 7(\Delta x)}{\Delta x}$

Find the limits in Exercises 23–26 using the continuity of $\sqrt{x}$.

23. $\lim\limits_{x \to 4} \dfrac{2x}{1 - \sqrt{x}}$ 24. $\lim\limits_{x \to 9} \dfrac{2x^2 - x}{\sqrt{x}}$

25. $\lim_{x \to 3}(1 - \sqrt{x})(2 + \sqrt{x})$ 26. $\lim_{x \to 2}(x^2 + 2x)\sqrt{x}$

Find the limits in Exercises 27–30 if they exist. Justify your answer.

27. $\lim\limits_{x \to 0}\left(\dfrac{x}{|x|} + 1\right)$ 28. $\lim\limits_{x \to 0}\left(x^2 + \dfrac{x}{|x|}\right)$

29. $\lim\limits_{x \to 1} \dfrac{|x - 1|}{x - 1}$ 30. $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$

Find the limits in Exercises 31–36 as $x \to \pm\infty$.

31. $\lim\limits_{x \to \infty} \dfrac{x - 1}{2x + 1}$ 32. $\lim\limits_{x \to \infty} \dfrac{2x^2 + 1}{3x^2 + 2}$

33. $\lim\limits_{x \to -\infty} \dfrac{2x - 1}{3x + 1}$ 34. $\lim\limits_{x \to -\infty} \dfrac{3x^3 + 2x^2 + 1}{4x^3 - x^2 + x + 2}$

35. $\lim\limits_{x \to \infty} \dfrac{|x|}{x}$ 36. $\lim\limits_{x \to -\infty} \dfrac{x}{|x|}$

37. For the function in Fig. 1.2.10, find $\lim_{x \to a} f(x)$ for $a = 0, 1, 2, 3, 4$ if it exists. In each case, tell whether $\lim_{x \to a} f(x) = f(a)$.

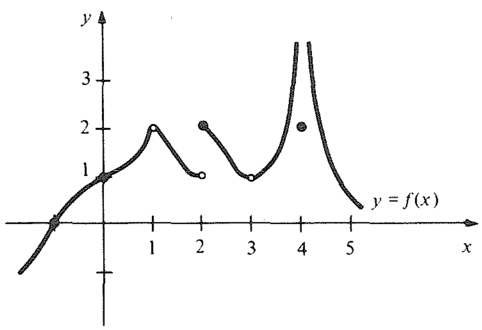


Figure 1.2.10. Find the limits at 0, 1, 2, 3, 4.

38. Find $\lim_{x \to a} f(x)$, where $a = -2$, 0, and 1 for f sketched in Fig. 1.2.11.

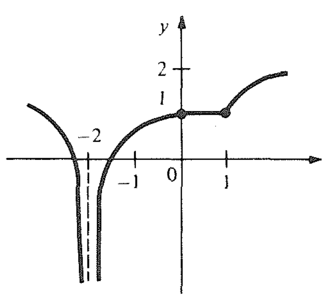


Figure 1.2.11. Find $\lim_{x \to a} f(x)$ at the indicated points.

Refer to Fig. 1.2.12 for Exercises 39 and 40 (assume that the functions keep going as they appear to).

39. Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

40. Find $\lim_{x \to \infty} g(x)$ and $\lim_{x \to -\infty} g(x)$.

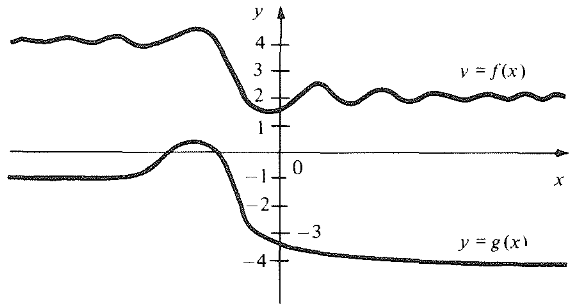


Figure 1.2.12. Find the limits at $\pm\infty$.

Find the limits in Exercises 41–44. If the limit is $\pm\infty$, give that as your answer.

41. $\lim\limits_{x \to 0}\left(-\dfrac{x}{x^3}\right)$ 42. $\lim\limits_{y \to 3} \dfrac{y - 4}{y^2 - 6y + 9}$

43. $\lim\limits_{x \to \sqrt{5}} \dfrac{2}{x^2 - 5}$ 44. $\lim\limits_{x \to 0} \dfrac{x^2 + 5x}{x^2}$

Find the limits in Exercises 45–58 if they exist.

45. $\lim\limits_{u \to 0} \dfrac{u^3 + 2u^2 + u}{u}$

46. $\lim\limits_{x \to \infty} \dfrac{x^3 + 2}{3x^3 + x}$

47. $\lim\limits_{x \to 2} \dfrac{2x}{(x - 2)^2}$

48. $\lim\limits_{x \to -2} \dfrac{x + 2}{|x + 2|}$

49. $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 6x + 8}$

50. $\lim\limits_{x \to 4} \dfrac{x^2 - 5x + 6}{x^2 - 6x + 8}$

51. $\lim\limits_{t \to 4} \dfrac{t^2 + 2\sqrt{t}}{|t|}$

52. $\lim\limits_{s \to \infty} \dfrac{s^2 - 2s + 1}{2s^2 + 3s + 2}$

53. $\lim\limits_{\Delta x \to 0} \dfrac{(5 + \Delta x)^3 - 5^3}{\Delta x}$

54. $\lim\limits_{\Delta x \to 0} \dfrac{(\Delta x)^4 + 2(\Delta x)^3 + 2\Delta x}{\Delta x}$

55. $\lim\limits_{x \to 1} \dfrac{3(x^3 - 1)}{x - 1}$

56. $\lim\limits_{q \to 3} \sqrt{|q - 3|}$

57. $\lim\limits_{s \to 3} \dfrac{3s^2 - 2s - 21}{(s - 3)^2}$

58. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x}}{x^2 + 1}$

59. How should $f(x) = (x^5 - 1)/(x - 1)$ be defined at $x = 1$ in order that $\lim_{x \to 1} f(x) = f(1)$?

60. How should $g(t) = (t^2 + 4t)/(t^2 - 4t)$ be defined at $t = 0$ to make $\lim_{t \to 0} g(t) = g(0)$?

★61. A block of ice melts in a room held at 75°F. Let $f(t)$ be the base area of the block and $g(t)$ the height of the block, measured with a ruler at time t.
   (a) Assume that the block of ice melts completely at time T. What values would you assign to $f(T)$ and $g(T)$?
   (b) Give physical reasons why $\lim_{t \to T} f(t) = f(T)$ and $\lim_{t \to T} g(t) = g(T)$ need not both hold. What are the limits?
   (c) The limiting volume of the ice block at time T is zero. Write this statement as a limit formula.
   (d) Using (b) and (c), illustrate the product rule for limits.

★62. A thermometer is stationed at x centimeters from a candle flame. Let $f(x)$ be the Celsius scale reading on the thermometer. Assume that the glass in the thermometer will crack upon contact with the flame.
   (a) Explain physically why $f(0)$ doesn't make any sense.
   (b) Describe in terms of the thermometer scale the meaning of $\lim_{x \to 0+} f(x)$ (i.e., the limit of $f(x)$ as x approaches zero through positive values).
   (c) Draw a realistic graph of $f(x)$ for a scale with maximum value 200°C. (Assume that the flame temperature is 400°C.)
   (d) Repeat (c) for a maximum scale value of 500°C.

★63. Suppose that $f(x) \neq 0$ for all $x \neq x_0$ and that $\lim_{x \to x_0} f(x) = \infty$. Can you conclude that $\lim_{x \to x_0}[1/f(x)] = 0$? Explain.

★64. Draw a figure, similar to Figs. 1.2.3 and 1.2.4, which illustrates the sum rule in our box on basic properties of limits.

★65. (a) Prove the extended sum rule in the box on derived properties of limits for the case $n = 4$ by using the basic sum rule and using the extended sum rule for the case $n = 3$ proved in the text.
   (b) Assume that the extended sum rule holds when $n = 16$; prove from your assumption that it holds when $n = 17$.
   (c) Assume that the extended sum rule holds for some given integer $n \geqslant 2$; prove that it holds for the integer $n + 1$.
   (d) According to the principle of induction, if a statement is true for $n + 1$ whenever it is true for n, and is true for some specific integer, m, then the statement is also true for $m + 1, m + 2, m + 3, \ldots,$ i.e., it is true for all integers larger than m. Use induction and the basic sum rule to prove the extended sum rule.

★66. Prove the extended product rule for limits by induction (see Exercise 65) and the basic properties of limits.

1.3 The Derivative as a Limit and the Leibniz Notation

The derivative is the limit of a difference quotient.

We are now ready to tie together the discussion of the derivative in Section 1.1 with the discussion of limits in Section 1.2.

Let $f(x)$ be a function such as the one graphed in Fig. 1.3.1. Recall the following items from Section 1.1: If $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ are two points on the graph, we write $\Delta y = f(x_0 + \Delta x) - f(x_0)$ and call $\Delta y/\Delta x$ the

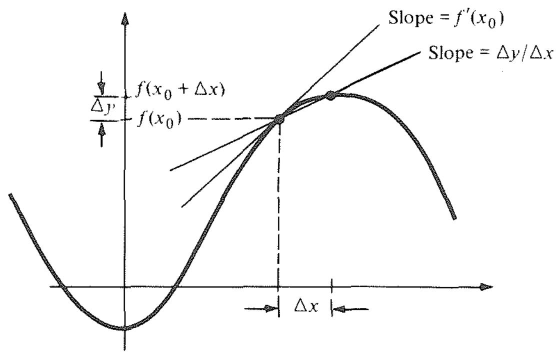


Figure 1.3.1. The limit of $\Delta y/\Delta x$ as $\Delta x \to 0$ is $f'(x_0)$.

difference quotient. This difference quotient is the slope of a secant line, as shown in the figure; moreover, if $f(x)$ is distance as a function of time, then $\Delta y/\Delta x$ is an average velocity. If Δx is small, then $\Delta y/\Delta x$ approximates the derivative $f'(x_0)$. Using these ideas, we were led to conclude that $f'(x_0)$ is the slope of the tangent line; moreover if $f(x)$ represents distance as a function of time, $f'(x_0)$ is the instantaneous velocity at time x_0. We can now make our discussion of $f'(x_0)$ more precise using the language of limits.

Suppose that the domain of a function $f(x)$ contains an open interval about a given number x_0. (For example, we might have $x_0 = 3$, and $f(x)$ might be defined for all x which obey $1 < x < 4$.) Consider the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

as a function of the variable Δx. The domain of the difference quotient then consists of those Δx, positive or negative, which are near enough to zero so that $f(x_0 + \Delta x)$ is defined. Since Δx appears in the denominator, $\Delta x = 0$ is not in the domain of the difference quotient. (For instance, in the example just mentioned with $x_0 = 3$ and $1 < x < 4$, $\Delta y/\Delta x$ would be defined for $-2 < \Delta x < 0$ and $0 < \Delta x < 1$.) As the examples in Section 1.1 indicated, we should look at the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$. This leads to the following definition of the derivative in terms of limits.

Formal Definition of the Derivative

Let $f(x)$ be a function whose domain contains an open interval about x_0. We say that f is *differentiable at* x_0 when the following limit exists:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x};$$

$f'(x_0)$ is then called the *derivative* of $f(x)$ at x_0.

Example 1 Suppose that $f(x) = x^2$. Then $f'(3) = 6$ by the quadratic function rule with $a = 1$, $b = 0 = c$ and $x_0 = 3$. Justify that $f'(3) = 6$ directly from the formal definition of the derivative and the rules for limits.

Solution We write the difference quotient and simplify:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(3 + \Delta x)^2 - 3^2}{\Delta x} = \frac{6\Delta x + (\Delta x)^2}{\Delta x}.$$

The independent variable is now Δx, but, of course, we can still use the rules for limits given in the previous section. By the replacement rule, we can cancel:

$$\lim_{\Delta x \to 0} \frac{6\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (6 + \Delta x),$$

provided the latter limit exists. However, $6 + \Delta x$ is a polynomial in the variable Δx and is defined at $\Delta x = 0$, so by the continuity of rational functions, $\lim_{\Delta x \to 0}(6 + \Delta x) = 6 + 0 = 6$. ▲

Example 2 Use the formal definition of the derivative and the rules for limits to differentiate x^3.

Solution Letting $f(x) = x^3$, we have

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^3 - x_0^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x_0^3 + 3x_0^2 \Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3 - x_0^3}{\Delta x} \quad \text{(expanding the cube)} \\ &= \lim_{\Delta x \to 0} \frac{3x_0^2 \Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(3x_0^2 + 3x_0 \Delta x + (\Delta x)^2\right) \qquad \text{(by the replacement rule)} \\ &= 3x_0^2 \end{aligned}$$

(using the continuity of rational functions and setting $\Delta x = 0$).

The derivative of x^3 at x_0 is therefore $3x_0^2$. ▲

As the next example shows, we can write x instead of x_0 when differentiating by the limit method, as long as we remember that x is to be held constant when we let $\Delta x \to 0$.

Example 3 If $f(x) = 1/x$, find $f'(x)$ for $x \neq 0$.

Solution The difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{1/(x + \Delta x) - 1/x}{\Delta x} = \frac{x - (x + \Delta x)}{x(x + \Delta x)\Delta x} = -\frac{\Delta x}{x(x + \Delta x)\Delta x}.$$

Here x is being held constant at some nonzero value, and $\Delta y/\Delta x$ is considered as a function of Δx. Note that Δx is in the domain of the difference quotient provided that $\Delta x \neq 0$ and $\Delta x \neq -x$.

For $\Delta x \neq 0$, $\Delta y/\Delta x$ equals $-1/x(x + \Delta x)$, so, by the replacement rule,

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \left(-\frac{1}{x(x + \Delta x)} \right) \\ &= -\frac{1}{x^2} \qquad \text{(by the continuity of rational functions).} \end{aligned}$$

Thus, $f'(x) = -1/x^2$. ▲

If we look back over the examples we have done, we may see a pattern. The derivative of x^3 is $3x^2$ by Example 2. The derivative of x^2 is given by the quadratic function rule as $2x^1 = 2x$. The derivative of $x = x^1$ is $1 \cdot x^0 = 1$, and the derivative of $1/x = x^{-1}$ is $(-1)x^{-2}$ by Example 3. In each case, when we differentiate x^n, we get nx^{n-1}. This general rule makes it unnecessary to memorize individual cases. In the next section, we will prove the rule for n a positive integer, and eventually we will prove it for all numbers n. For now, let us see how to prove the rule for $x^{1/2} = \sqrt{x}$. We should get $\frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = 1/2\sqrt{x}$.

Example 4 Differentiate $\sqrt{x}$ $(x > 0)$.

Solution The difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

In order to cancel Δx, we perform a trick: rationalize by multiplying numerator and denominator by $\sqrt{x+\Delta x}+\sqrt{x}$:

$$\frac{\Delta y}{\Delta x}=\frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{(\Delta x)(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\frac{(x+\Delta x)-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}.$$

Notice that this trick enabled us to cancel Δx in the numerator and denominator.

Now recall from the previous section that $\lim_{x\to x_0}\sqrt{x}=\sqrt{x_0}$. Thus, $\lim_{\Delta x\to 0}\sqrt{x+\Delta x}=\sqrt{x}$. Hence, by the quotient rule for limits,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{\lim_{\Delta x\to 0}(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\frac{1}{\lim_{\Delta x\to 0}\sqrt{x+\Delta x}+\lim_{\Delta x\to 0}\sqrt{x}}\qquad\text{(sum rule)}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\qquad\text{(continuity of }\sqrt{x}\text{)}.$$

Thus, the derivative is indeed $1/2\sqrt{x}$. ▲

Next, let us establish a general relationship between differentiability and continuity.

Theorem: Differentiability Implies Continuity

If $f'(x_0)$ exists, then f is continuous at x_0; that is, $\lim_{x\to x_0}f(x)=f(x_0)$.

Proof We first note that $\lim_{x\to x_0}f(x)=f(x_0)$ is the same as $\lim_{x\to x_0}(f(x)-f(x_0))=0$ (by the sum rule and then the constant function rule applied to the constant $f(x_0)$). With $\Delta x=x-x_0$, and $\Delta y=f(x_0+\Delta x)-f(x_0)$, this is, in turn, the same as $\lim_{\Delta x\to 0}\Delta y=0$. Now we use again the trick of multiplying numerator and denominator by an appropriate factor:

$$\lim_{\Delta x\to 0}\Delta y=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\cdot\Delta x\right)\qquad\text{(replacement rule)}$$

$$=\left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\right)\left(\lim_{\Delta x\to 0}\Delta x\right)\qquad\text{(product rule)}$$

$$=f'(x_0)\cdot 0\qquad\left(\text{since }\lim_{\Delta x\to 0}\Delta x=0\right)$$

$$=0.$$

This proves our claim. ■

The converse theorem is not true; the following is a counterexample.

Example 5 Show that $f(x)=|x|$ has no derivative at $x_0=0$, yet is continuous. (See Section R.2 for a review of absolute values.)

Solution The difference quotient at $x_0 = 0$ is $(|0 + \Delta x| - |0|)/\Delta x = |\Delta x|/\Delta x$, which is 1 for $\Delta x > 0$ and -1 for $\Delta x < 0$. As we saw in Example 7 of Section 1.2, the function $|\Delta x|/\Delta x$ has no limit at $\Delta x = 0$, so the derivative of $|x|$ at $x_0 = 0$ does not exist.

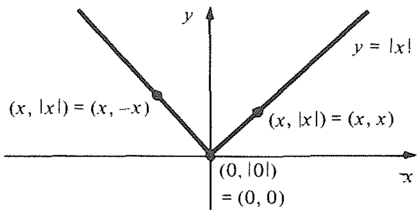


Figure 1.3.2. As $x \to 0$ from either direction, $|x| \to 0$, so $f(x) = |x|$ is continuous at 0.

On the other hand, as $x \to 0$, $f(x) = |x| \to 0$ as well (see Fig. 1.3.2), so $\lim_{x\to 0}|x| = |0|$; $|x|$ is continuous at 0. ▲

We have seen that the derivative $f'(x_0)$ of $y = f(x)$ at x_0 is approximated by the difference quotient $\Delta y/\Delta x$, where $\Delta x = x - x_0$.

In the view of Gottfried Wilhelm von Leibniz (1646–1716), one of the founders of calculus, one could think of Δx as becoming "infinitesimal." The resulting quantity he denoted as dx, the letters d and Δ being the Roman and Greek equivalents of one another. When Δx became the infinitesimal dx, Δy simultaneously became the infinitesimal dy and the ratio $\Delta y/\Delta x$ became dy/dx, which was no longer an approximation to the derivative but exactly equal to it. The notation dy/dx has proved to be extremely convenient—not as a ratio of infinitesimal quantities but as a *synonym* for $f'(x)$.[2]

Leibniz Notation

If $y = f(x)$, the derivative $f'(x)$ may be written

$$\frac{dy}{dx},\quad dy/dx,\quad \frac{df(x)}{dx},\quad (d/dx)\,f(x),\quad \text{or}\quad d(f(x))/dx.$$

This is just a notation and does not represent division. If we wish to denote the value $f'(x_0)$ of f' at a specific point x_0, we may write

$$\left.\frac{dy}{dx}\right|_{x_0} \quad \text{and} \quad \left.\frac{df(x)}{dx}\right|_{x_0}.$$

dy/dx is read "the derivative of y with respect to x" or "dy by dx."

Of course, we can use this notation if the variables are named other than x and y. For instance, the area A of a square of side l is $A = l^2$ so we can write $dA/dl = 2l$.

In the f' notation, if $f(x) = 3x^2 + 2x$, then $f'(x) = 6x + 2$. Using the Leibniz notation we may write:

$$\text{if}\quad y = 3x^2 + 2x, \qquad \text{then}\quad \frac{dy}{dx} = 6x + 2.$$

[2] Modern developments in mathematics have made it possible to give rigorous definitions of dx and dy. The earlier objections to infinitesimals as quantities which were supposed to be smaller than any real number but still nonzero have been circumvented through the work of the logician Abraham Robinson (1918–1974). A calculus textbook based upon this approach is H. J. Keisler, *Elementary Calculus*, Prindle, Weber, and Schmidt, Boston (1976).

We can also use the even more compact notation

$$\frac{d(3x^2+2x)}{dx} = 6x+2 \quad \text{or} \quad \frac{d}{dx}(3x^2+2x) = 6x+2.$$

Here the d/dx may be thought of as a symbol for the *operation* of differentiation. It takes the place of the prime (′) in the functional notation.

Example 6 (a) Find the slope m of the graph $y=\sqrt{x}$ at $x=4$. (b) Find the velocity v of a bus whose distance function is t^3.

Solution (a) The slope is a derivative. The derivative of $\sqrt{x}$ is $dy/dx = d(\sqrt{x})/dx = 1/(2\sqrt{x})$ by Example 4. Evaluating at $x=4$ gives $m = 1/(2\sqrt{4}) = \frac{1}{4}$.
(b) $v = (d/dt)(t^3) = 3t^2$. ▲

Supplement to Section 1.3
Filling a Pond

We conclude this section with a harder and perhaps more interesting application that previews some important topics to be considered in detail later: rates of change (Section 2.1) and integration (Chapter 4).

Suppose that a mountain brook swells from a trickle to a torrent each year as the snows melt. At the time t (days after midnight on March 31), the flow rate is known to be $3t^2$ thousand liters per day. We wish to build a large pond which holds the runoff for the entire month of April. How big must the pond be?

The main difficulty here is that a flow rate of, say $3\cdot(5)^2$ at midnight of April 5 does not tell us directly how much water will be in the pond on April 5, but merely how fast water will be pouring in at that moment. Let's see if we can somehow handle that difficulty.

Designate the unknown amount of water in the pond at time t by $A=f(t)$. During a short time interval Δt starting at t, the amount of water entering the pond will be at least $3t^2\Delta t$ and no more than $3(t+\Delta t)^2\Delta t$. Thus, $\Delta A = A(t+\Delta t) - A(t)$ is slightly larger than $3t^2\,\Delta t$. For Δt very small, we can presumably take $\Delta A \approx 3t^2\Delta t$, i.e., $\Delta A/\Delta t \approx 3t^2$. However, for Δt very small, $\Delta A/\Delta t$ approximates the derivative dA/dt. Thus our problem becomes the following. Find the "amount" function $f(t)$, given that the derivative obeys $f'(t) = 3t^2$.

Now, turning Example 2 around, we know one function which obeys $f'(t)=3t^2$, namely $f(t)=t^3$. This solution is reasonable in the sense that $f(0)=0$, i.e., the pond is empty at midnight of March 31. Could there also be a different amount function that works? Not really. If a capacity of t^3 thousand liters is exactly right to accommodate all the influx up to time t, no other capacity will be exactly right. We thus have our answer: at midnight on April 30, $A=f(30)=(30)^3$; our pond must hold 27,000 thousand liters.

Exercises for Section 1.3

Use the formal definition of the derivative and the rules for limits to find the derivatives of the functions in Exercises 1–12.

1. $f(x) = x^2 + x$
2. $f(x) = 2x^2 - 3x$
3. $f(x) = 5x^3$
4. $f(x) = 2x^3$
5. $f(x) = \dfrac{3}{x}$
6. $f(x) = \dfrac{10}{x}$
7. $f(x) = x^2 + \dfrac{3}{x}$
8. $f(x) = x^3 - \dfrac{2}{x}$
9. $f(x) = 2\sqrt{x}$
10. $f(x) = 8\sqrt{x}$
11. $f(x) = 2x^2 - \sqrt{x} + \dfrac{1}{x}$
12. $f(x) = x^3 + 2x - \dfrac{1}{\sqrt{x}}$

Show that the functions in Exercises 13 and 14 have no derivative at x_0, yet are continuous.

13. $f(x) = 1 + |x|;\ x_0 = 0$
14. $g(x) = |x + 1|;\ x_0 = -1$.

Find dy/dx in the Exercises 15–18.

15. $y = x^2 - x$
16. $y = x - 5x^2$
17. $y = 3x^3 + x$
18. $y = x^2 - x^3$

In Exercises 19 and 20, find the slope of the line tangent to the given graph at the given point.

19. $y = 8\sqrt{x}\,;\ x_0 = 9$
20. $y = 2x^2 - \sqrt{x} + 1/x;\ x_0 = 1$

In Exercises 21 and 22, $f(t)$ is the position of a car on a straight road at time t. Find its velocity at the given time.

21. $f(t) = 5t^3;\ t = 1$
22. $f(t) = t^2 - t^3;\ t = \frac{1}{2}$

In Exercises 23–26, evaluate the derivatives.

23. $\dfrac{d(3/t)}{dt}$ at $t = 1$
24. $\dfrac{d}{dt}\left[t^3 + 2t - \dfrac{1}{\sqrt{t}}\right]$ at $t = 2$
25. $\left.\dfrac{d}{dx}(3x^3 + x)\right|_{x=1}$
26. $\left.\dfrac{d}{dx}\left(\dfrac{10}{x}\right)\right|_{x=2}$
27. Using the limit method, find the derivative of $2x^3 + x^2 - 3$ at $x_0 = 1$.
28. (a) Expand $(a + b)^4$. (b) Use the limit method to differentiate x^4.

Use limits to find the derivatives of the functions in Exercises 29–32.

29. $f(x) = 1/x^2$
30. $\sqrt[3]{x}$
31. $f(x) = (x^2 + x)/2x$
32. $f(x) = x/(1 + x^2)$

★33. Find an example of a function which is continuous everywhere and which is differentiable everywhere except at *two* points.

★34. (a) Show by the quadratic function rule that if $f(x) = ax^2 + bx + c$, $g(x) = dx^2 + ex + f$, and $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$; i.e., $(d/dx)[f(x) + g(x)] = (d/dx)\,f(x) + (d/dx)\,g(x)$.

(b) Show from the rules for limits that if $f(x)$ and $g(x)$ are differentiable functions, then
$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
and
$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

(c) Argue geometrically, using graphs and slopes, that a function $C(x)$ for which $C'(x) = 0$ must be a constant function.

(d) Combining (b) and (c), show that if $f'(x) = g'(x)$, then there is some constant C such that $f(x) = g(x) + C$. Illustrate your result graphically.

(e) In (d) show that if $f(0) = 0 = g(0)$, then $f(x) = g(x)$ for all x.

(f) Use (e) to argue that in the pond example discussed in the Supplement, $A(t) = t^3$ is the only appropriate solution of $A'(t) = 3t^2$.

★35. (a) Do some calculator experiments to guess $\lim_{x\to0}(\sin x/x)$ and $\lim_{x\to0}[(1 - \cos x)/x]$, where the angle x is measured in radians.

(b) *Given* the facts that $\lim_{x\to0}(\sin x/x) = 1$ and $\lim_{x\to0}[(1 - \cos x)/x] = 0$, use trigonometric identities to show:
$$\frac{d(\sin x)}{dx} = \cos x,$$
$$\frac{d(\cos x)}{dx} = -\sin x.$$

★36. Suppose that the mountain brook in the Supplement has a flow rate of $t^2/12 + 2t$ thousand liters per day t days after midnight on March 31. What is the runoff for the first 15 days of April? The entire month?

1.4 Differentiating Polynomials

Polynomials can be differentiated using the power rule, the sum rule, and the constant multiple rule.

In Section 1.3, we learned how to compute derivatives of some simple functions using limits. Now we shall use the limit method to find a general rule for differentiating polynomials like $f(x) = 3x^5 - 8x^4 + 4x + 2$. To do this systematically, we shall break apart a polynomial using two basic operations.

First, we recognize that a polynomial is a sum of monomials: for example, $f(x) = 3x^5 - 8x^4 + 4x + 2$ is the sum of $3x^5$, $-8x^4$, $4x$ and 2. Second, a monomial is a product of a constant and a power of x. For example, $3x^5$ is the product of 3 and x^5.

Let us work backward, starting with powers of x. Thus our first goal is to differentiate x^n, where n is a positive integer. We have already seen that for $n = 1, 2$, or 3 (as well as $n = -1$ or $\frac{1}{2}$), the derivative of x^n is nx^{n-1}.

We can establish this rule for any positive integer n by using limits. Let $f(x) = x^n$. To compute $f'(x)$, we must find the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Now $f(x) = (x + \Delta x)^n = (x + \Delta x)(x + \Delta x) \ldots (x + \Delta x)$, n times. To expand a product like this, we select one term from each factor, multiply these n terms, and then add all such products. For example,

$$\begin{aligned}(x + \Delta x)(x + \Delta x) &= x^2 + x\,\Delta x + (\Delta x)x + (\Delta x)^2\\ &= x^2 + 2x\,\Delta x + (\Delta x)^2,\\ (x + \Delta x)(x + \Delta x)(x + \Delta x) &= x^3 + x^2\Delta x + x(\Delta x)x + (\Delta x)x^2 + (\Delta x)^2 x\\ &\quad + x(\Delta x)^2 + (\Delta x)x(\Delta x) + (\Delta x)^3\\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.\end{aligned}$$

For $(x + \Delta x)^n$, notice that the coefficient of Δx will be nx^{n-1} since there will be exactly n terms which contain $n - 1$ factors of x and one of Δx. Thus

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \big(\text{terms involving } (\Delta x)^2, (\Delta x)^3, \ldots, (\Delta x)^n\big).$$

If you are familiar with the binomial theorem, you will know the remaining terms; however, their exact form is not needed here. For $\Delta x \neq 0$, dividing out Δx now gives

$$\begin{aligned}\frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{(x + \Delta x)^n - x^n}{\Delta x}\\ &= \frac{nx^{n-1}\Delta x + \big(\text{terms involving } (\Delta x)^2, \ldots, (\Delta x)^n\big)}{\Delta x}\\ &= nx^{n-1} + \big(\text{terms involving } (\Delta x), \ldots, (\Delta x)^{n-1}\big).\end{aligned}$$

The terms involving $\Delta x, \ldots, (\Delta x)^{n-1}$ add up to a polynomial in Δx, so the limit as $\Delta x \to 0$ is obtained by setting $\Delta x = 0$ and by using the continuity of rational functions (Section 1.2). Therefore,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = nx^{n-1}.$$

Power Rule

To differentiate a power x^n, bring down the exponent as a factor and then reduce the exponent by 1.

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$; that is

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad n = 1, 2, 3, \ldots .$$

Example 1 Compute the derivatives of x^8, x^{12}, and x^{99}.

Solution $(d/dx)x^8 = 8x^7$, $(d/dx)x^{12} = 12x^{11}$, and $(d/dx)x^{99} = 99x^{98}$. ▲

Next, we consider the *constant multiple rule*, stated in the following box.

Constant Multiple Rule

To differentiate the product of a number k with $f(x)$, multiply the number k by the derivative $f'(x)$:

$$(kf)'(x) = kf'(x),$$

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x).$$

Proof of the Constant Multiple Rule Let $h(x) = kf(x)$. By the definition of the derivative and the basic properties of limits, we get

$$\begin{aligned} h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{kf(x+\Delta x) - kf(x)}{\Delta x} = \lim_{\Delta x \to 0} k\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right) \\ &= \left(\lim_{\Delta x \to 0} k\right) \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] = kf'(x). \blacksquare \end{aligned}$$

Example 2 Differentiate

(a) $-3x^7$ (b) $5\sqrt{x}$ (c) $\dfrac{8}{x}$ and (d) $-6ax^2$.

Solution (a) By the constant multiple and power rules,

$$\frac{d}{dx}(-3x^7) = (-3)\frac{d}{dx}x^7 = (-3)(7)x^6 = -21x^6.$$

(b) From Example 4, Section 1.3, $(d/dx)\sqrt{x} = 1/2\sqrt{x}$. Thus, by the constant multiple rule,

$$\frac{d}{dx}(5\sqrt{x}) = 5\frac{d}{dx}\sqrt{x} = \frac{5}{2\sqrt{x}}.$$

(c) By Example 3, Section 1.3, $(d/dx)(1/x) = -1/x^2$. Thus

$$\frac{d}{dx}\left(\frac{8}{x}\right) = \frac{-8}{x^2}.$$

(d) Although it is not explicitly stated, we assume that a is constant (letters from the beginning of the alphabet are often used for constants). Thus, by the constant multiple rule

$$\frac{d}{dx}(-6ax^2) = -6a\frac{d}{dx}x^2 = -12ax. \blacktriangle$$

The final basic technique we need is the *sum rule*.

If f and g are two functions, the sum $f+g$ is defined by the formula $(f+g)(x) = f(x) + g(x)$.

Example 3 Let $f(x) = 3x^2 + 5x + 9$ and $g(x) = 2x^2 + 5x$. Use the quadratic function rule to verify that $(f+g)' = f' + g'$.

Solution By the quadratic function rule, $f'(x) = 6x + 5$ and $g'(x) = 4x + 5$, thus $f'(x) + g'(x) = 10x + 10$. On the other hand, $f(x) + g(x) = 5x^2 + 10x + 9$, so $(f+g)'(x) = 10x + 10 = f'(x) + g'(x)$. $\blacktriangle$

Sum Rule

To differentiate a sum, take the sum of the derivatives:

$$(f+g)' = f' + g'$$

or

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

or

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

To be convinced that a mathematical statement such as the sum rule is true, one should ideally do three things:

1. Check some simple examples directly.
2. Have a mathematical justification (proof).
3. Have a simple physical model, application, or diagram that makes the result plausible.

In Example 3 we checked the sum rule in a simple case. In the next paragraph we give a mathematical justification for the sum rule. In the Supplement at the end of the section, we give a simple physical model.

Proof of the Sum Rule By the definition of the derivative as a limit, $(f+g)'(x_0)$ is equal to

$$\lim_{\Delta x \to 0} \frac{(f+g)(x_0+\Delta x) - (f+g)(x_0)}{\Delta x}$$

(if this limit exists). We can rewrite the limit as

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) + g(x_0+\Delta x) - f(x_0) - g(x_0)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} \right].$$

By the sum rule for *limits*, this is

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}.$$

If f and g are differentiable at x_0, these two limits are just $f'(x_0)$ and $g'(x_0)$. Thus $f+g$ is differentiable at x_0, and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$. ■

The sum rule extends to several summands. For example, to find a formula for the derivative of $f(x)+g(x)+h(x)$, we apply the sum rule twice:

$$\frac{d}{dx}[f(x)+g(x)+h(x)] = \frac{d}{dx}[f(x) + (g(x)+h(x))]$$
$$= \frac{d}{dx}f(x) + \frac{d}{dx}(g(x)+h(x))$$
$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) + \frac{d}{dx}h(x).$$

Example 4 Find the formula for the derivative $8f(x) - 10g(x)$.

Solution We use the sum and constant multiple rules:

$$\frac{d}{dx}[8f(x) - 10g(x)] = \frac{d}{dx}[8f(x)] + \frac{d}{dx}[-10g(x)]$$
$$= 8\frac{d}{dx}f(x) - 10\frac{d}{dx}g(x). \blacktriangle$$

Now we can differentiate any polynomial.

Example 5 Differentiate (a) $(x^{95} + x^{23} + 2x^2 + 4x + 1)$; (b) $4x^9 - 6x^5 + 3x$.

Solution (a) $\frac{d}{dx}(x^{95} + x^{23} + 2x^2 + 4x + 1)$

$$= \frac{d}{dx}(x^{95}) + \frac{d}{dx}(x^{23}) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(1)$$

(sum rule)

$$= 95x^{94} + 23x^{22} + 4x + 4$$

(power rule and constant multiple rule).

(b) $\frac{d}{dx}(4x^9 - 6x^5 + 3x) = 36x^8 - 30x^4 + 3. \blacktriangle$

Here, for reference, is a general rule, but you need not memorize it, since you can readily do any example by using the sum, power, and constant multiple rules.

Derivative of a Polynomial

If $f(x) = c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0$, then

$$f'(x) = nc_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + 2c_2 x + c_1.$$

Example 6 Find the derivative of $x^3 + 5x^2 - 9x + 2$.

Solution
$$\frac{d}{dx}(x^3 + 5x^2 - 9x + 2) = \frac{d}{dx}x^3 + \frac{d}{dx}(5x^2) - \frac{d}{dx}(9x) + \frac{d}{dx}2$$
$$= 3x^2 + 10x - 9. \blacktriangle$$

Example 7 (a) Compute $f'(s)$ if $f(s) = (s^2 + 3)(s^3 + 2s + 1)$.

(b) Find $\frac{d}{dx}(10x^3 - 8/x + 5\sqrt{x})$.

Solution (a) First we expand the product

$$(s^2 + 3)(s^3 + 2s + 1) = (s^5 + 2s^3 + s^2) + (3s^3 + 6s + 3)$$
$$= s^5 + 5s^3 + s^2 + 6s + 3.$$

Now we differentiate this polynomial:

$$f'(s) = 5s^4 + 15s^2 + 2s + 6.$$

(b) $$\frac{d}{dx}\left(10x^3 - \frac{8}{x} + 5\sqrt{x}\right) = \frac{d}{dx}(10x^3) + \frac{d}{dx}\left(\frac{-8}{x}\right) + \frac{d}{dx}(5\sqrt{x})$$
$$= 10\frac{d}{dx}(x^3) - 8\frac{d}{dx}(x^{-1}) + 5\frac{d}{dx}(x^{1/2})$$
$$= 30x^2 + \frac{8}{x^2} + \frac{5}{2}x^{-1/2} = 30x^2 + \frac{8}{x^2} + \frac{5}{2\sqrt{x}}. \blacktriangle$$

The differentiation rules we have learned can be applied to the problems of finding slopes and velocities.

Example 8 (a) Find the slope of the tangent line to the graph of

$$y = x^4 - 2x^3 + 1 \quad \text{at} \quad x = 1.$$

(b) A train has position $x = 3t^2 + 2 - \sqrt{t}$ at time t. Find the velocity of the train at $t = 2$.

Solution (a) The slope is the derivative at $x = 1$. The derivative is

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^3 + 1) = 4x^3 - 6x^2.$$

At $x = 1$, this is $4 \cdot 1^3 - 6 \cdot 1^2 = -2$, the required slope.

(b) The velocity is

$$\frac{dx}{dt} = \frac{d}{dt}(3t^2 + 2 - \sqrt{t}) = 6t - \frac{1}{2\sqrt{t}}.$$

At $t = 2$, we get

$$\left.\frac{dx}{dt}\right|_2 = 6 \cdot 2 - \frac{1}{2\sqrt{2}} = 12 - \frac{1}{2\sqrt{2}} = \frac{24\sqrt{2} - 1}{2\sqrt{2}} = \frac{48 - \sqrt{2}}{4}. \quad \blacktriangle$$

Supplement to Section 1.4
A Physical Model for the Sum Rule

Imagine a train, on a straight track, whose distance at time x from a fixed reference point on the ground is $f(x)$. There is a runner on the train whose distance from a reference point on the train is $g(x)$. Then the distance of the runner from the fixed reference point on the ground is $f(x) + g(x)$. (See Fig. 1.4.1.) Suppose that, at a certain time x_0, the runner is going at 20 kilometers

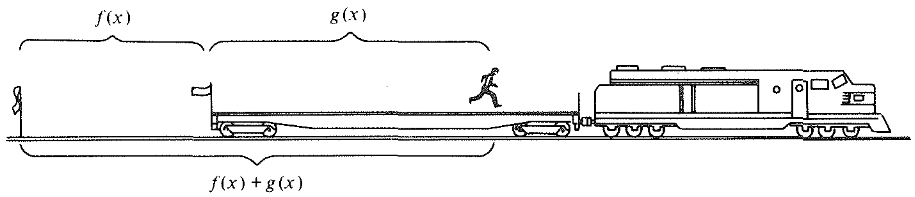


Figure 1.4.1. The sum rule illustrated in terms of velocities.

per hour with respect to the train while the train is going at 140 kilometers per hour—that is, $f'(x_0) = 140$ and $g'(x_0) = 20$. What is the velocity of the runner as seen from an observer on the ground? It is the sum of 140 and 20—that is, 160 kilometers per hour. Considered as the sum of two velocities, the number 160 is $f'(x_0) + g'(x_0)$; considered as the velocity of the runner with respect to the ground, the number 160 is $(f + g)'(x_0)$. Thus we have $f'(x_0) + g'(x_0) = (f + g)'(x_0)$.[3]

[3] The fact that one does not add velocities this way in the theory of special relativity does not violate the sum rule. In classical mechanics, velocities are derivatives, but in relativity, velocities are not simply derivatives, so the formula for their combination is more complicated.

Exercises for Section 1.4

Differentiate the functions in Exercises 1–12.

1. x^{10}
2. x^{14}
3. x^{33}
4. x^5
5. $-5x^4$
6. $-53x^{20}$
7. $3x^{10}$
8. $8x^{100}$
9. $3\sqrt{x}$
10. $2/x$
11. $-8\sqrt{x}$
12. $-6/x$

In Exercises 13–16, verify the sum rule for the given pair of functions.

13. $f(x) = 3x^2 + 6$, $g(x) = x + 7$
14. $f(x) = 8x + 9$, $g(x) = x^2 - 1$
15. $f(x) = x^2 + x + 1$, $g(x) = -1$
16. $f(x) = 2x^2 - 3x + 6$, $g(x) = -x^2 + 8x - 9$
17. Find a formula for the derivative of $f(x) - 2g(x)$.
18. Find a formula for the derivative of $3f(x) + 2g(x)$.

Differentiate the functions in Exercises 19–22.

19. $x^5 + 8x$
20. $5x^3$
21. $t^5 + 6t^2 + 8t + 2$
22. $s^{10} + 8s^9 + 5s^8 + 2$

Differentiate the functions in Exercises 23–34.

23. $f(x) = x^4 - 7x^2 - 3x + 1$
24. $h(x) = 3x^{11} + 8x^5 - 9x^3 - x$
25. $g(s) = s^{13} + 12s^8 - \frac{3}{8}s^7 + s^4 + \frac{4}{3}s^3$
26. $f(y) = -y^3 - 8y^2 - 14y - \frac{1}{3}$
27. $f(x) = x^4 - 3x^3 + 2x^2$
28. $f(t) = t^4 + 4t^3$
29. $g(h) = 8h^{10} + h^9 - 56.5h^2$
30. $h(\gamma) = \pi\gamma^{10} + \frac{21}{7}\gamma^9 - \sqrt{2}\,\gamma^2$
31. $p(x) = (x^2 + 1)^3$
32. $r(t) = (t^4 + 2t^2)^2$
33. $f(t) = (t^3 - 17t + 9)(3t^5 - t^2 - 1)$
34. $h(x) = (x^4 - 1)(x^2 + x + 2)$
35. Find $f'(r)$ if $f(r) = -5r^6 + 5r^4 - 13r^2 + 15$.
36. Find $g'(s)$ if $g(s) = s^7 + 13s^6 - 18s^3 + \frac{3}{2}s^2$.
37. Find $h'(t)$ if $h(t) = (t^4 + 9)(t^3 - t)$.
38. Differentiate $x^5 + 2x^4 + 7$.
39. Differentiate $(u^4 + 5)(u^3 + 7u^2 + 19)$.
40. Differentiate $(3t^5 + 9t^3 + 5t)(t + 1)$.

Differentiate the functions in Exercises 41–46.

41. $f(x) = x^2 - \sqrt{x}$
42. $f(x) = 3x^3 + \dfrac{1}{x}$
43. $f(x) = x^3 - 2x + 2\sqrt{x}$
44. $f(x) = x^5 - \sqrt{x} + \dfrac{1}{x}$
45. $f(x) = (1 - \sqrt{x})(1 + \sqrt{x})$
46. $f(x) = (1 + \sqrt{x})\sqrt{x}$
47. A particle moves on a line with position $f(t) = 16t^2 + (0.03)t^4$ at time t. Find the velocity at $t = 8$.
48. Suppose that the position x of a car at time t is $(t - 2)^3$.
    (a) What is the velocity at $t = -1, 0, 1$?
    (b) Show that the average velocity over every interval of time is positive.
    (c) There is a stop sign at $x = 0$. A police officer gives the driver a ticket because there was no period of time during which the car was stopped. The driver argues that, since his velocity was zero at $t = 2$, he obeyed the stop sign. Who is right?
49. Find the slope of the tangent line to the graph of $x^4 - x^2 + 3x$ at $x = 1$.
50. Find the slope of the line tangent to the graph of $f(x) = x^8 + 2x^2 + 1$ at $(1, 4)$.

For each of the functions in Exercises 51–54, find a function whose derivative is $f(x)$. (Do not find $f'(x)$.)

51. $f(x) = x^2$
52. $f(x) = x^2 + 2x + 3$
53. $f(x) = x^n$ (n any positive integer)
54. $f(x) = (x + 3)(x^2 + 1)$
55. Verify the constant multiple rule for general quadratic functions, i.e., show that $(kf)'(x) = kf'(x)$ if $f(x) = ax^2 + bx + c$.
56. Verify the sum rule for general cubic functions using the formula for the derivative of a polynomial.
57. Let $V(r)$ be the volume V of a sphere as a function of the radius r. Show that $V'(r)$ is the surface area.
58. Let $V(l)$ be the volume of a cube as a function of l, where $2l$ is length of one of its edges. Show that $V'(l)$ is the surface area.

★59. Explain the constant multiple rule in terms of a change of units in distance from miles to kilometers.

★60. Show that if two polynomials have the same derivative, they must differ by a constant.

1.5 Products and Quotients

To differentiate a product, differentiate each factor in turn and sum the results.

We have given general rules for the derivative of a sum and a constant multiple. We now turn to products and quotients.

The product fg and quotient f/g of two functions are defined by $(fg)(x) = f(x)g(x)$ and $(f/g)(x) = f(x)/g(x)$, the latter being defined only when $g(x) \neq 0$. The formulas for $(fg)'$ and $(f/g)'$ are more complicated than those for $(f+g)'$ and $(kf)'$, but they are just as straightforward to apply. Before developing the correct formulas, let us convince ourselves that $(fg)'$ is *not* $f'g'$.

Example 1 Let $f(x) = x^2$ and $g(x) = x^3$. Is $(fg)'$ equal to $f'g'$?

Solution Notice that the product function is obtained simply by multiplying the formulas for f and g: $(fg)(x) = (x^2)(x^3) = x^5$. Thus, $(fg)'(x) = 5x^4$. On the other hand, $f'(x) = 2x$ and $g'(x) = 3x^2$, so $(f'g')(x) = f'(x)g'(x) = 6x^3$. Since $5x^4$ and $6x^3$ are not the same function, $(fg)'$ is *not* equal to $f'g'$. ▲

Example 1 shows that the derivative of the product of two functions is not the product of their derivatives. We state the correct rule for products now and discuss below why it is true.

Product Rule

To differentiate a product $f(x)g(x)$, differentiate each factor and multiply it by the other one, then add the two products:

$$(fg)'(x) = f(x)\,g'(x) + f'(x)\,g(x)$$

or

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

Example 2 (a) Verify the product rule for f and g in Example 1.
(b) Verify the product rule for $f(x) = x^m$ and $g(x) = x^n$, where m and n are natural numbers.

Solution (a) We know that $(fg)'(x) = 5x^4$. On the other hand, $f(x)g'(x) + f'(x)g(x) = (x^2)(3x^2) + (2x)(x^3) = 5x^4$, so the product rule gives the right answer.
(b) By the power rule in Section 1.4, $f'(x) = mx^{m-1}$ and $g'(x) = nx^{n-1}$, so that $(fg)'(x) = f(x)g'(x) + f'(x)g(x) = x^m(nx^{n-1}) + (mx^{m-1})x^n = (n+m)x^{m+n-1}$. On the other hand, $(fg)(x) = x^m x^n = x^{m+n}$, so again by the power rule $(fg)'(x) = (m+n)x^{m+n-1}$, which checks. ▲

The form of the product rule may be a surprise to you. Why should that strange combination of f, g, and their derivatives be the derivative of fg? The following mathematical justification should convince you that the product rule is correct.

Proof of the Product Rule

To find $(fg)'(x_0)$, we take the limit

$$\lim_{\Delta x \to 0} \frac{(fg)(x_0 + \Delta x) - (fg)(x_0)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x)g(x_0 + \Delta x) - f(x_0)g(x_0)}{\Delta x}. \tag{1}$$

Simplifying this expression is not as straightforward as for the sum rule. We may make use of a geometric device: think of $f(x)$ and $g(x)$ as the lengths of the sides of a rectangle; then $f(x)g(x)$ is its area. The rectangles for $x = x_0$ and $x = x_0 + \Delta x$ are shown in Fig. 1.5.1. The area of the large rectangle is

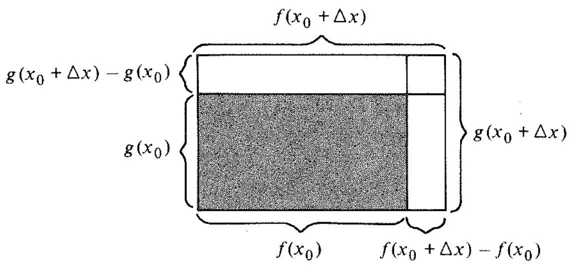


Figure 1.5.1 The geometry behind the proof of the product rule.

$f(x_0 + \Delta x)g(x_0 + \Delta x)$; that of the darker rectangle is $f(x_0)g(x_0)$. The difference $f(x_0 + \Delta x)g(x_0 + \Delta x) - f(x_0)g(x_0)$ is the area of the lighter region, which can be decomposed into three rectangles having areas

$$[f(x_0 + \Delta x) - f(x_0)]\, g(x_0),$$
$$[f(x_0 + \Delta x) - f(x_0)][g(x_0 + \Delta x) - g(x)],$$

and

$$f(x_0)[g(x_0 + \Delta x) - g(x)].$$

Thus we have the identity:

$$f(x_0 + \Delta x)\, g(x_0 + \Delta x) - f(x_0)\, g(x_0)$$
$$= [f(x_0 + \Delta x) - f(x_0)]\, g(x_0) + f(x_0)[g(x_0 + \Delta x) - g(x_0)]$$
$$+ [f(x_0 + \Delta x) - f(x_0)][g(x_0 + \Delta x) - g(x_0)]. \tag{2}$$

(If you do not like geometric arguments, you can verify this identity algebraically.)

Substituting (2) into (1), we obtain

$$\lim_{\Delta x \to 0} \left\{ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\, g(x_0) + f(x_0)\, \frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} + \frac{[f(x_0 + \Delta x) - f(x_0)][g(x_0 + \Delta x) - g(x_0)]}{\Delta x} \right\}. \tag{3}$$

By the sum and constant multiple rules for limits, (3) equals

$$\left[\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right] g(x_0) + f(x_0) \left[\lim_{\Delta x \to 0} \frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} \right]$$
$$+ \lim_{\Delta x \to 0} \frac{[f(x_0 + \Delta x) - f(x_0)][g(x_0 + \Delta x) - g(x_0)]}{\Delta x}. \tag{4}$$

We recognize the first two limits in (4) as $f'(x_0)$ and $g'(x_0)$, so the first two terms give $f'(x_0)g(x_0) + f(x_0)g'(x_0)$—precisely the product rule. To show that the third limit, represented geometrically by the small rectangle in the upper right-hand corner of Fig. 1.5.1, is zero, we use continuity of g (see Section 1.3). The product rule for *limits* yields

$$\lim_{\Delta x\to 0}\left[\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right]\cdot\lim_{\Delta x\to 0}\left[g(x_0+\Delta x)-g(x_0)\right]$$
$$=f'(x_0)\cdot 0=0. \blacksquare$$

Example 3 Using the product rule, differentiate $(x^2+2x-1)(x^3-4x^2)$. Check your answer by multiplying out first.

Solution

$$\frac{d}{dx}\left[(x^2+2x-1)(x^3-4x^2)\right]$$
$$=\frac{d(x^2+2x-1)}{dx}(x^3-4x^2)+(x^2+2x-1)\frac{d(x^3-4x^2)}{dx}$$
$$=(2x+2)(x^3-4x^2)+(x^2+2x-1)(3x^2-8x)$$
$$=(2x^4-6x^3-8x^2)+(3x^4-2x^3-19x^2+8x)$$
$$=5x^4-8x^3-27x^2+8x.$$

Multiplying out first,

$$(x^2+2x-1)(x^3-4x^2)=x^5-4x^4+2x^4-8x^3-x^3+4x^2$$
$$=x^5-2x^4-9x^3+4x^2.$$

The derivative of this is $5x^4-8x^3-27x^2+8x$, so our answer checks. ▲

Example 4 Differentiate $x^{3/2}$ by writing $x^{3/2}=x\cdot\sqrt{x}$ and using the product rule.

Solution We know that $(d/dx)x=1$ and $(d/dx)\sqrt{x}=1/(2\sqrt{x})$. Thus, the product rule gives

$$\frac{d}{dx}(x^{3/2})=\frac{d}{dx}(x\cdot\sqrt{x})=\left(\frac{d}{dx}x\right)\sqrt{x}+x\frac{d}{dx}\sqrt{x}$$
$$=\sqrt{x}+x\cdot\frac{1}{2\sqrt{x}}=\sqrt{x}+\frac{1}{2}\sqrt{x}=\frac{3}{2}\sqrt{x}.$$

This result may be written $(d/dx)x^{3/2}=\frac{3}{2}x^{1/2}$, which is another instance of the rule $(d/dx)(x^n)=nx^{n-1}$ for noninteger n. ▲

We now come to quotients. Let $h(x)=f(x)/g(x)$, where f and g are differentiable at x_0, and suppose $g(x_0)\neq 0$ so that the quotient is defined at x_0. If we *assume* the existence of $h'(x_0)$, it is easy to compute its value from the product rule.

Since $h(x)=f(x)/g(x)$, we have $f(x)=g(x)h(x)$. Apply the product rule to obtain

$$f'(x_0)=g'(x_0)h(x_0)+g(x_0)h'(x_0).$$

Solving for $h'(x_0)$, we get

$$h'(x_0)=\frac{f'(x_0)-g'(x_0)h(x_0)}{g(x_0)}=\frac{f'(x_0)-g'(x_0)\left[f(x_0)/g(x_0)\right]}{g(x_0)}$$

$$= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}.$$

This is the quotient rule.

Quotient Rule

To differentiate a quotient $f(x)/g(x)$ (where $g(x) \neq 0$), take the derivative of the numerator times the denominator, subtract the numerator times the derivative of the denominator, and divide the result by the square of the denominator:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad \text{or}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(du/dx)v - u(dv/dx)}{v^2}.$$

When you use the quotient rule, it is important to remember which term in the numerator comes first. (In the product rule, both terms occur with a plus sign, so the order does not matter.) One memory aid is the following: Write your guess for the right formula and set $g = 1$ and $g' = 0$. Your formula should reduce to f'. If it comes out as $-f'$ instead, you have the terms in the wrong order.

Example 5 Differentiate $\dfrac{x^2}{x^3+5}$.

Solution By the quotient rule, with $f(x) = x^2$ and $g(x) = x^3 + 5$,

$$\frac{d}{dx}\left(\frac{x^2}{x^3+5}\right) = \frac{2x(x^3+5) - x^2(3x^2)}{(x^3+5)^2}$$

$$= \frac{x}{(x^3+5)^2}(2x^3 + 10 - 3x^3) = \frac{x(-x^3+10)}{(x^3+5)^2}. \blacktriangle$$

Example 6 Find the derivative of (a) $h(x) = (2x+1)/(x^2-2)$ and (b) $\sqrt{x}/(1+3x^2)$.

Solution (a) By the quotient rule with $f(x) = 2x + 1$ and $g(x) = x^2 - 2$,

$$h'(x) = \frac{2(x^2-2) - (2x+1)2x}{(x^2-2)^2} = \frac{2x^2 - 4 - 4x^2 - 2x}{(x^2-2)^2}$$

$$= -\frac{2x^2 + 2x + 4}{(x^2-2)^2}.$$

(b) $$\frac{d}{dx}\left(\frac{\sqrt{x}}{1+3x^2}\right) = \frac{(1+3x^2)\cdot(1/2\sqrt{x}) - \sqrt{x}\cdot 6x}{(1+3x^2)^2} = \frac{1-9x^2}{2\sqrt{x}\,(1+3x^2)^2}. \blacktriangle$$

In the argument given for the quotient rule, we assumed that $h'(x_0)$ exists; however, we can prove the quotient rule more carefully by the method of limits.

Proof of the Quotient Rule The derivative of $h(x) = f(x)/g(x)$ *at* x_0 *is given by the following limit*:

$$\begin{aligned} h'(x_0) &= \lim_{\Delta x \to 0} \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x)/g(x_0 + \Delta x) - f(x_0)/g(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x)g(x_0) - f(x_0)g(x_0 + \Delta x)}{g(x_0)g(x_0 + \Delta x)\Delta x}. \end{aligned}$$

A look at the calculations in the limit derivation of the product rule suggests that we add $-f(x_0)g(x_0) + f(x_0)g(x_0) = 0$ to the numerator. We get

$$\begin{aligned} h'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_0 + \Delta x)}{g(x_0)g(x_0 + \Delta x)\Delta x} \\ &= \lim_{\Delta x \to 0} \left\{ \frac{1}{g(x_0)g(x_0 + \Delta x)} \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} g(x_0) \right.\right. \\ &\qquad \left.\left. - f(x_0) \frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} \right] \right\} \\ &= \frac{1}{\lim_{\Delta x \to 0} g(x_0 + \Delta x)} \frac{1}{g(x_0)} \left[f'(x_0)g(x_0) - f(x_0)g'(x_0) \right]. \end{aligned} \tag{5}$$

Since g is differentiable at x_0, it is continuous there (see Section 1.3), and so

$$\lim_{\Delta x \to 0} g(x_0 + \Delta x) = g(x_0). \tag{6}$$

Substituting (6) into (5) gives the quotient rule. ■

Certain special cases of the quotient rule are particularly useful. If $f(x) = 1$, then $h(x) = 1/g(x)$ and we get the reciprocal rule:

Reciprocal Rule

To differentiate the reciprocal $1/g(x)$ of a function (where $g(x) \neq 0$), take the negative of the derivative of the function and divide by the square of the function:

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{[g(x)]^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}.$$

Example 7 Differentiate (a) $1/(x^3 + 3x^2)$ and (b) $1/(\sqrt{x} + 2)$.

Solution (a)

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x^3 + 3x^2}\right) &= -\frac{1}{(x^3 + 3x^2)^2} \frac{d}{dx}(x^3 + 3x^2) \\ &= -\frac{3x^2 + 6x}{(x^3 + 3x^2)^2}. \end{aligned}$$

(b) $$\frac{d}{dx}\frac{1}{(\sqrt{x}+2)} = -\frac{1}{(\sqrt{x}+2)^2}\frac{d}{dx}(\sqrt{x}+2)$$
$$= -\frac{1}{2\sqrt{x}(\sqrt{x}+2)^2}. \blacktriangle$$

Combining the reciprocal rule with the power rule from Section 1.4 enables us to differentiate negative powers.[4] If k is a positive integer, then

$$\frac{d}{dx}(x^{-k}) = \frac{d}{dx}\left(\frac{1}{x^k}\right) = -\frac{1}{(x^k)^2}\frac{d}{dx}(x^k)$$
$$= -\frac{1}{x^{2k}}(kx^{k-1}) = -kx^{-k-1}.$$

Writing n for the negative integer $-k$, we have $(d/dx)(x^n) = nx^{n-1}$, just as for positive n. Recalling that $(d/dx)(x^0) = (d/dx)(1) = 0$, we have established the following general rule.

Integer Power Rule

If n is any (positive, negative or zero) integer, $(d/dx)x^n = nx^{n-1}$. (When $n < 0$, x must be unequal to zero.)

Example 8 Differentiate $1/x^6$

Solution $(d/dx)(1/x^6) = (d/dx)(x^{-6}) = -6x^{-6-1} = -6x^{-7}$. ▲

We conclude this section with a summary of the differentiation rules obtained so far. Some of these rules are special cases of the others. For instance, the linear and quadratic function rules are special cases of the polynomial rule, and the reciprocal rule is the quotient rule for $f(x) = 1$. Remember that the basic idea for differentiating a complicated function is to break it into its component parts and combine the derivatives of the parts according to the rules.

Example 9 Differentiate (a) $3x^4 + \frac{2}{x} - \frac{5}{x^3}$ and (b) $\frac{1}{(x^2+3)(x^2+4)}$.

Solution (a) By the sum, power, and constant multiple rules,

$$\frac{d}{dx}\left(3x^4 + \frac{2}{x} - \frac{5}{x^3}\right) = 3\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^{-1}) - 5\frac{d}{dx}(x^{-3})$$
$$= 3\cdot 4x^3 + 2(-1)x^{-2} - 5(-3)x^{-4}$$
$$= 12x^3 - \frac{2}{x^2} + \frac{15}{x^4}.$$

(b) Let $f(x) = (x^2+3)(x^2+4)$. By the product rule, $f'(x) = 2x(x^2+4) + (x^2+3)2x = 4x^3 + 14x$. By the reciprocal rule, the derivative of $1/f(x)$ is

$$-\frac{f'(x)}{[f(x)]^2} = -\frac{4x^3+14x}{(x^2+3)^2(x^2+4)^2}. \blacktriangle$$

[4] Students requiring a review of negative exponents should read Section R.3.

Example 10 We derived the reciprocal rule from the quotient rule. By writing $f(x)/g(x) = f(x)\cdot[1/g(x)]$, show that the quotient rule also follows from the product rule and the reciprocal rule.

Solution

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \left(f\cdot\frac{1}{g}\right)'(x)\\ &= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(\frac{1}{g}\right)'(x)\\ &= \frac{f'(x)}{g(x)} + f(x)\cdot\frac{-g'(x)}{[g(x)]^2}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.\end{aligned}$$

(This calculation gives another way to reconstruct the quotient rule if you forget it—assuming, of course, that you remembered the reciprocal rule.) ▲

Differentiation Rules

	The derivative of	*is*	*In Leibniz notation*
Linear function	$bx + c$	b	$\frac{d}{dx}(bx+c) = b$
Quadratic function	$ax^2 + bx + c$	$2ax + b$	$\frac{d}{dx}(ax^2+bx+c) = 2ax + b$
Sum	$f(x) + g(x)$	$f'(x) + g'(x)$	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Constant multiple	$kf(x)$	$kf'(x)$	$\frac{d}{dx}(ku) = k\frac{du}{dx}$
Power	x^n $\{n$ any integer$\}$	nx^{n-1}	$\frac{d}{dx}(x^n) = nx^{n-1}$
Polynomial	$c_nx^n + \cdots + c_2x^2 + c_1x + c_0$	$nc_nx^{n-1} + \cdots + 2c_2x + c_1$	$\frac{d}{dx}(c_nx^n + \cdots + c_2x^2 + c_1x + c_0) = nc_nx^{n-1} + \cdots + 2c_2x + c_1$
Product	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$	$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$
Quotient	$f(x)/g(x)$ $\{g(x)\neq 0\}$	$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(du/dx)v - u(dv/dx)}{v^2}$
Reciprocal	$1/g(x)$ $\{g(x)\neq 0\}$	$-g'(x)/[g(x)]^2$	$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}$

Exercises for Section 1.5

Compute the derivatives of the functions in Exercises 1–12 by using the product rule. Verify your answer by multiplying out and differentiating the resulting polynomials.

1. $(x^2+2)(x+8)$
2. $(x+1)(x-1)$
3. $(x^4+x)(x^3-2)$
4. $(x^2+3x)(2x-1)$
5. $(x^2+2x+1)(x-1)$
6. $(x^3+3x^2+3x+1)(x-1)$
7. $(x^2+2x+2)(x^2+3x)$
8. $(x^2+4x+8)(x^2+2x-1)$
9. $(x-1)(x^2+x+1)$
10. $(x-2)(x^2+2x+1)$
11. $(x-1)(x^3+x^2+x+1)$
12. $(x^3+2)(x^2+2x+1)$

In Exercises 13–16, differentiate the given function by writing it as indicated and using the product rule.

13. $x^{5/2}=x^2\cdot\sqrt{x}$
14. $x=\sqrt{x}\cdot\sqrt{x}$
15. $x^{7/2}=x^3\cdot\sqrt{x}$
16. $x^2=\sqrt{x}\cdot x^{3/2}$

Differentiate the functions in Exercises 17–30.

17. $\dfrac{x-2}{x^2+3}$
18. $\dfrac{x^3-3x+5}{x^4-1}$
19. $\dfrac{x^7-x^2}{x^3+1}$
20. $\dfrac{5x^3+x-10}{3x^4+2}$
21. $\dfrac{x^2+2}{x^2-2}$
22. $\dfrac{x}{1-x^2}$
23. $\dfrac{1}{x^2}+\dfrac{x}{x^2+1}$
24. $\dfrac{1}{t^9}$
25. $\dfrac{r^2+2}{r^8}$
26. $\dfrac{x}{2}+\dfrac{3}{x+1}$
27. $\left(\dfrac{s}{1-s}\right)^2$
28. $\dfrac{(x^3-1)^2}{x^3+1}$
29. $\dfrac{(x^2+1)^2+1}{(x^2+1)^2-1}$
30. $\dfrac{4}{(x^2-1)(x+7)^2}$

Find the indicated derivatives in Exercises 31–38.

31. $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right)$
32. $\dfrac{d}{dx}\left(\dfrac{1}{x^5+5x^2}\right)$
33. $\dfrac{d}{dx}\left(\dfrac{1}{(x+1)^2}\right)$
34. $\dfrac{d}{dx}\left(\dfrac{1}{(x^2+9)^2}\right)$
35. $\dfrac{d}{ds}(s^3+1)(s^5+1)$
36. $\dfrac{d}{du}(u^4+2u)(u^3+2u)$
37. $\dfrac{d}{dy}\left[4(y+1)^2-2(y+1)-\dfrac{1}{y+1}\right]$
38. $\dfrac{d}{dz}\left[\dfrac{(z-3)^2+2}{(z-3)^2+3}\right]$

Differentiate the functions in Exercises 39–46.

39. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$
40. $\dfrac{1}{(\sqrt{x}+1)^2}$
41. $(3\sqrt{x}+1)x^2$
42. $\dfrac{1}{x+\sqrt{x}}$
43. $\dfrac{x}{1+\sqrt{x}}$
44. $\dfrac{1-\sqrt{x}}{x^2+2}$
45. $\dfrac{\sqrt{2}}{1+3\sqrt{x}}$
46. $\dfrac{8x^{3/2}+\sqrt{x}}{1-\sqrt{x}}$
47. Use the reciprocal rule twice to differentiate $1/[1/g(x)]$ and show that the result is $g'(x)$.
48. Differentiate x^m/x^n by the quotient rule and compare your answer with the derivative of x^{m-n} obtained by the power rule.
49. Find the slope of the line tangent to the graph of $f(x)=1/\sqrt{x}$ at $x=2$.
50. Find the slope of the line tangent to the graph of $f(x)=(2x+1)/(3x+1)$ at $x=1$.

Let $f(x)=4x^5-13x$ and $g(x)=x^3+2x-1$. Find the derivatives of the functions in Exercises 51–56.

51. $f(x)g(x)$
52. $[f(x)+x^3-7x][g(x)]$
53. $xf(x)+g(x)$
54. $\dfrac{f(x)}{g(x)}+(x^3-3x)-7$
55. $\dfrac{g(x)}{f(x)}$
56. $\dfrac{f(x)}{f(x)+g(x)-4x^5-x^3+10x+1}$

★57. Let $P(x)$ be a quadratic polynomial. Show that $(d/dx)(1/P(x))$ is zero for at most one value of x in its domain. Find an example of $P(x)$ for which $(d/dx)(1/P(x))$ is never zero on its domain.

★58. Calculate the following limits by expressing each one as the derivative of some function:

(a) $\lim\limits_{x\to 1}\dfrac{x^8-x^7+3x^2-3}{x-1}$,

(b) $\lim\limits_{x\to 2}\dfrac{1/x^3-1/2^3}{x-2}$,

(c) $\lim\limits_{x\to -1}\dfrac{x^2+x}{(x+2)(x+1)}$.

1.6 The Linear Approximation and Tangent Lines

A good approximation to $f(x_0 + \Delta x)$ *is* $f(x_0) + f'(x_0)\Delta x$.

In Section 1.1, we saw that the derivative $f'(x_0)$ is the slope of the tangent line to the graph $y = f(x)$. This section explores the relationship between the graph of f and its tangent line a little further.

Recall from Section R.4 that the equation of the straight line through (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0).$$

In particular we get the following formula for the tangent line to $y = f(x)$ (see Figure 1.6.1).

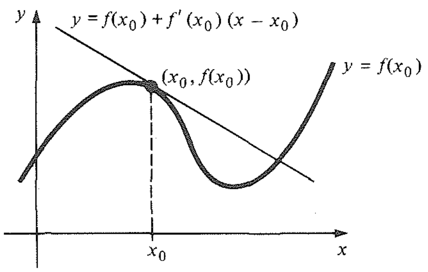


Figure 1.6.1. The tangent line to the graph $y = f(x)$ at $(x_0, f(x_0))$.

Equation of the Tangent Line

The equation of the line tangent to $y = f(x)$ at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Example 1 (a) Find the equation of the line tangent to the graph $y = \sqrt{x} + 1/(2(x+1))$ at $x = 1$.

(b) Find the equation of the line tangent to the graph of the function $f(x) = (2x+1)/(3x+1)$ at $x = 1$.

Solution (a) Here $x_0 = 1$ and $f(x) = \sqrt{x} + 1/2(x+1)$. We compute

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2(x+1)^2},$$

so $f'(1) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$.

Since $f(1) = 1 + \frac{1}{4} = \frac{5}{4}$, the tangent line has equation $y = \frac{5}{4} + \frac{3}{8}(x-1)$, i.e., $8y = 3x + 7$.

(b) By the quotient rule, $f'(x)$ equals $[2(3x+1) - (2x+1)3]/(3x+1)^2 = -1/(3x+1)^2$. The equation of the tangent line is

$$y = f(1) + f'(1)(x-1) = \tfrac{3}{4} - \tfrac{1}{16}(x-1)$$

or $y = -\frac{1}{16}x + \frac{13}{16}$. ▲

Example 2 Where does the line tangent to $y=\sqrt{x}$ at $x=2$ cross the x axis?

Solution Here $dy/dx = 1/2\sqrt{x}$, which equals $1/2\sqrt{2}$ at $x=2$. Since $y=\sqrt{2}$ at $x=2$, the equation of the tangent line is

$$y=\sqrt{2}+\frac{1}{2\sqrt{2}}(x-2).$$

This line crosses the x axis when $y=0$ or $0=\sqrt{2}+(1/2\sqrt{2})(x-2)$. Solving for x, we get $x=-2$. Thus the tangent line crosses the x axis at $x=-2$. ▲

We have used the idea of limit to pass from difference quotients to derivatives. We can also go in the other direction: given $f(x_0)$ and $f'(x_0)$, we can use the derivative to get an approximate value for $f(x)$ when x is near x_0.

According to the definition of the derivative, the difference quotient $\Delta y/\Delta x = [f(x_0+\Delta x)-f(x_0)]/\Delta x$ is close to $f'(x_0)$ when Δx is small. That is, the *d*ifference

$$\frac{\Delta y}{\Delta x}-f'(x_0)=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}-f'(x_0)=d$$

is small when Δx is small. Multiplying the preceding equation by Δx and rearranging gives

$$f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x+d\Delta x. \tag{1}$$

Suppose now that we know $f(x_0)$ and $f'(x_0)$ and that we wish to evaluate f at the nearby point $x=x_0+\Delta x$. Formula (1) expresses $f(x)$ as a sum of three terms, the third of which becomes small—even compared to Δx—as $\Delta x\to 0$. By dropping this term, we obtain the approximation

$$f(x)\approx f(x_0)+f'(x_0)\Delta x. \tag{2}$$

In terms of $x=x_0+\Delta x$, we have

$$f(x)\approx f(x_0)+f'(x_0)(x-x_0). \tag{3}$$

The right-hand side of (3) is a linear function of x, called the *linear approximation* to f at x_0. Notice that its graph is just the tangent line to the graph of f at $(x_0, f(x_0))$. (See Fig. 1.6.2.)

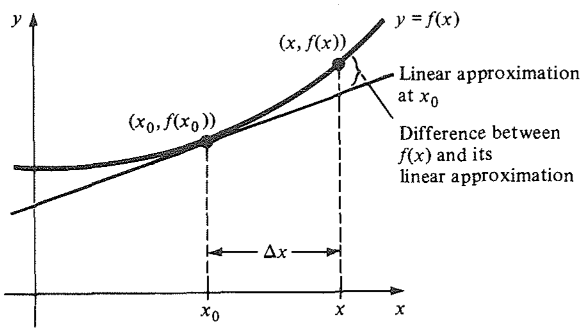


Figure 1.6.2. As Δx approaches zero, the difference between $f(x)$ and the approximation $f(x_0)+f'(x_0)(x-x_0)$ becomes arbitrarily small compared to $\Delta x = x - x_0$.

The linear approximation is also called the *first-order* approximation. Second-order and higher-order approximations are introduced in Section 12.5.

Example 3 (a) Show that the linear approximation to $(x_0+\Delta x)^2$ is $x_0^2+2x_0\Delta x$. (b) Calculate an approximate value for $(1.03)^2$. Compare with the actual value. Do the same for $(1.0003)^2$ and $(1.0000003)^2$.

Solution (a) Let $f(x)=x^2$, so $f'(x)=2x$. Thus the linear approximation to $f(x_0+\Delta x)$ is $f(x_0)+f'(x_0)\Delta x = x_0^2+2x_0\Delta x$.
(b) Let $x_0=1$ and $\Delta x=0.03$; from (a), the approximate value is $1+2\Delta x=1.06$. The exact value is 1.0609. If $\Delta x=0.0003$, the approximate value is 1.0006 (very easy to compute), while the exact value is 1.00060009 (slightly harder to compute). If $\Delta x=0.0000003$, the approximate value is 1.0000006, while the exact value is 1.00000060000009. Notice that the error decreases even faster than Δx. ▲

The Linear Approximation

For x near x_0, $f(x_0)+f'(x_0)(x-x_0)$ is a good approximation for $f(x)$.

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x \quad \text{or} \quad \Delta y\approx f'(x_0)\Delta x$$

The error becomes arbitrarily small, compared with Δx, as $\Delta x\to 0$.

Example 4 Calculate an approximate value for the following quantities using the linear approximation around $x_0=9$. Compare with the values on your calculator.

(a) $\sqrt{9.02}$ (b) $\sqrt{10}$ (c) $\sqrt{8.82}$ (d) $\sqrt{8}$

Solution Let $f(x)=\sqrt{x}$ and recall that $f'(x)=1/2\sqrt{x}$. Thus the linear approximation is

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x,$$

i.e.,

$$\sqrt{x_0+\Delta x}\approx\sqrt{x_0}+\frac{1}{2\sqrt{x_0}}\Delta x.$$

(a) Let $x_0=9$ and $\Delta x=0.02$, so $x_0+\Delta x=9.02$. Thus

$$\sqrt{9.02}\approx\sqrt{9}+\frac{1}{2\sqrt{9}}0.02=3+\frac{0.02}{6}=3.0033\ldots .$$

On our calculator we get 3.0033315.
(b) Let $x_0=9$ and $\Delta x=1$; then

$$\sqrt{10}\approx\sqrt{9}+\frac{1}{2\sqrt{9}}1=3+\frac{1}{6}=3.166\ldots .$$

On our calculator we get 3.1622777.
(c) Let $x_0=9$ and $\Delta x=-0.18$; then

$$\sqrt{8.82}\approx\sqrt{9}+\frac{1}{2\sqrt{9}}(-0.18)$$
$$=3+\frac{1}{6}(-0.18)=3-0.03=2.97.$$

On our calculator we get 2.9698485.

(d) Let $x_0 = 9$ and $\Delta x = -1$; then

$$\sqrt{8} = \sqrt{9} - \frac{1}{2\sqrt{9}} = 3 - \frac{1}{6} = 2.833\ldots.$$

On our calculator, we get 2.8284271.

Notice that the linear approximation gives the best answers in (a) and (c), where Δx is smallest. ▲

Example 5 Calculate the linear approximation to the area of a square whose side is 2.01. Draw a geometric figure, obtained from a square of side 2, whose area is exactly that given by the linear approximation.

Solution $A = f(r) = r^2$. The linear approximation near $r_0 = 2$ is given by $f(r_0) + f'(r_0)(r - r_0) = r_0^2 + 2r_0(r - r_0) = 4 + 4(r - r_0)$. When $r - r_0 = 0.01$, this is 4.04.

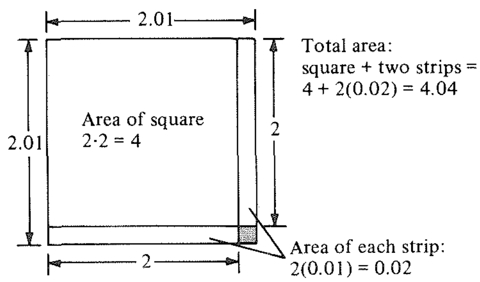


Figure 1.6.3. The linear approximation to the change in area with respect to a side has error equal to the shaded area. (Diagram not to scale.)

The required figure is shown in Fig. 1.6.3. It differs from the square of side 2.01 only by the small shaded square in the corner, whose area is $(0.01)^2 = 0.0001$. ▲

Example 6 Calculate an approximate value for

$$\frac{2}{\sqrt{0.99} + (0.99)^2}$$

and compare with the numerical value on your calculator.

Solution We let $f(x) = 2/(\sqrt{x} + x^2)$ and note that we are asked to calculate $f(1 - 0.01)$. By the linear approximation,

$$f(1 - 0.01) \approx f(1) - f'(1)(0.01).$$

Note that $f(1) = 1$. We calculate $f'(x)$ by the quotient rule:

$$f'(x) = -\frac{2\left(1/2\sqrt{x} + 2x\right)}{\left(\sqrt{x} + x^2\right)^2}$$

$$= -\frac{1 + 4x\sqrt{x}}{\sqrt{x}\left(\sqrt{x} + x^2\right)^2}.$$

At $x = 1$, $f'(1) = -\frac{5}{4}$, so

$$f(1 - 0.01) \approx 1 + \tfrac{5}{4}(0.01) = 1.0125.$$

On our calculator we find $f(0.99) = 1.0126134$, in rather good agreement. ▲

Exercises for Section 1.6

In Exercises 1–4, find the equation of the line tangent to the graph of the given function at the indicated point and sketch:

1. $y = 1 - x^2$; $x_0 = 1$
2. $y = x^2 - x$; $x_0 = 0$
3. $y = x^2 - 2x + 1$; $x_0 = 2$
4. $y = 3x^2 + 1 - x$; $x_0 = 5$

Find the equation of the tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$ in Exercises 5–8.

5. $f(x) = (x^2 - 7)\dfrac{3x}{x+2}$; $x_0 = 0$
6. $f(x) = \dfrac{1}{(x^2+4)^2}$; $x_0 = 0$
7. $f(x) = \left[\dfrac{1}{x} - 2x\right](x^2 + 2)$; $x_0 = \frac{1}{2}$
8. $f(x) = \dfrac{x^2}{1+x^2}$; $x_0 = 1$

In Exercises 9–12, find where the tangent line to the graph of the given function at the given point crosses the x axis.

9. $y = \dfrac{x}{x+1}$; $x_0 = 1$
10. $y = \dfrac{\sqrt{x}}{x+1}$; $x_0 = 2$
11. $y = \dfrac{2}{1+\sqrt{x}}$; $x_0 = 4$
12. $y = x(\sqrt{x} + 1)$; $x_0 = 1$

Calculate an approximate value for each of the squares in Exercises 13–16 and compare with the exact value:

13. $(2.02)^2$
14. $(199)^2$
15. $(4.999)^2$
16. $(-1.002)^2$

In Exercises 17–20, calculate an approximate value for the square root using the linear approximation at $x_0 = 16$. Compare with the value on your calculator.

17. $\sqrt{16.016}$
18. $\sqrt{17}$
19. $\sqrt{15.92}$
20. $\sqrt{15}$

Using the linear approximation, find an approximate value for the quantities in Exercises 21–24.

21. $(2.94)^4$
22. $(1.03)^4$
23. $(3.99)^3$
24. $(101)^8$
25. The radius of a circle is increased from 3 to 3.04. Using the linear approximation, what do you find to be the increase in the area of the circle?
26. The radius r of the base of a right circular cylinder of fixed height h is changed from 4 to 3.96. Using the linear approximation, approximate the change in volume V.
27. A sphere is increased in radius from 5 to 5.01. Using the linear approximation, estimate the increase in surface area (the surface area of a sphere of radius r is $4\pi r^2$).
28. Redo Exercise 27 replacing surface area by volume (the volume inside a sphere of radius r is $\frac{4}{3}\pi r^3$).

Calculate approximate values in Exercises 29–32.

29. $(x^2 + 3)(x + 2)$ if $x = 3.023$
30. $\dfrac{x^2}{x^3+2}$ if $x = 2.004$
31. $\dfrac{1}{(2.01)^2 + (2.01)^3}$
32. $\dfrac{1}{(1.99)^2 + (1.99)^4}$
33. Find the equation of the line tangent to the graph of $f(x) = x^8 + 2x^2 + 1$ at $(1, 4)$.
34. Find the equation of the tangent line to the graph of $x^4 - x^2 + 3x$ at $x = 1$.
35. Find the linear approximation for $1/0.98$.
36. Find the linear approximation for $1/1.98$.

Calculate approximate values for each of the quantities in Exercises 37–40.

37. $s^4 - 5s^3 + 3s - 4$; $s = 0.9997$
38. $\dfrac{x^4}{x^5 - 2x^2 - 1}$; $x = 2.0041$
39. $(2.01)^{20}$
40. $\dfrac{1}{(1.99)^2}$
41. Let $h(t) = -4t^2 + 7t + \frac{3}{4}$. Use the linear approximation to approximate values for $h(3.001)$, $h(1.97)$, and $h(4.03)$.
42. Let $f(x) = 3x^2 - 4x + 7$. Using the linear approximation, find approximate values for $f(2.02)$, $f(1.98)$, and $f(2.004)$. Compute the actual values without using a calculator and compare with the approximations. Compare the amount of time you spend in computing the approximations with the time spent in obtaining the actual values.
43. Let $g(x) = -4x^2 + 8x + 13$. Find $g'(3)$. Show that the linear approximation to $g(3 + \Delta x)$ always gives an answer which is too large, regardless of whether Δx is positive or negative. Interpret your answer geometrically by drawing a graph of g and its tangent line when $x_0 = 3$.
44. Let $f(x) = 3x^2 - 4x + 7$. Show that the linear approximation to $f(2 + \Delta x)$ always gives an answer which is too small, regardless of whether Δx is positive or negative. Interpret your answer geometrically by drawing a graph of f and its tangent line at $x_0 = 2$.

★45. Let $f(x) = x^4$.
  (a) Find the linear approximation to $f(x)$ near $x = 2$.
  (b) Is the linear approximation larger or smaller than the actual value of the function?
  (c) Find the largest interval containing $x = 2$ such that the linear approximattion is accurate within 10% when x is in the interval.

★46. (a) Give numerical examples to show that linear approximations to $f(x) = x^3$ may be either too large or too small.

(b) Illustrate your examples by sketching a graph of $y = x^3$, using calculated values of the function.

47. Show that a good approximation to $1/(1+x)$ when x is small is $1 - x$.

★48. If you travel 1 mile in $60 + x$ seconds, show that a good approximation to your average speed, for x small, is $60 - x$ miles per hour. (This works quite well on roads which have mileposts.) Find the error in this approximation if $x = 1, -1, 5, -5, 10, -10$.

49. Return to Exercise 47. Show experimentally that a better approximation to $1/(1+x)$ is $1 - x + x^2$. Use this result to refine the speedometer checking rule in Exercise 48.

★50. Devise a speedometer checking rule for metric units which works for speeds in the vicinity of about 90 or 100 kilometers per hour.

Review Exercises for Chapter 1

Differentiate the functions in Exercises 1–20.

1. $f(x) = x^2 - 1$
2. $f(x) = 3x^2 + 2x - 10$
3. $f(x) = x^3 + 1$
4. $f(x) = x^4 - 8$
5. $f(x) = 2x - 1$
6. $f(x) = 8x + 1$
7. $f(s) = s^2 + 2s$
8. $f(r) = r^4 + 10r + 2$
9. $f(x) = -10x^5 + 8x^3$
10. $f(x) = 12x^3 + 2x^2 + 2x - 8$
11. $f(x) = (x^2 - 1)(x^2 + 1)$
12. $f(x) = (x^3 + 2x + 3)(x^2 + 2)$
13. $f(x) = 3x^3 - 2\sqrt{x}$
14. $f(x) = x^4 + 9\sqrt{x}$
15. $f(x) = x^{50} + \dfrac{1}{x}$
16. $f(x) = x^9 - \dfrac{8}{x}$
17. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
18. $f(x) = \dfrac{\sqrt{x} + 2}{x^2 - 1}$
19. $f(x) = \dfrac{1}{\sqrt{x}\,(x^2 + 2)}$
20. $f(x) = \dfrac{\sqrt{x}}{(x^2 + 2)^2}$

Find the derivatives indicated in Exercises 21–30.

21. $\dfrac{d}{ds}(s+1)^2(\sqrt{s} + 2)$
22. $\dfrac{d}{du}\dfrac{u^2 + 2 + 3\sqrt{u}}{\sqrt{u}}$
23. $\dfrac{d}{dr}\dfrac{\pi r^2}{1 + \sqrt{r}}$
24. $\dfrac{d}{dv}\dfrac{\sqrt{3}\,v + 1}{\sqrt{v} + 2}$
25. $\dfrac{d}{dt}(3t^2 + 2)^{-1}$
26. $\dfrac{d}{dx}\dfrac{3}{x^3 + 2x + 1}$
27. $\dfrac{d}{dp}\dfrac{\sqrt{2}\,p^2}{p^2 + 1}$
28. $\dfrac{d}{dq}(q + 2)^{-3}$
29. $\dfrac{d}{dx}\dfrac{1}{\sqrt{x}\,(\sqrt{x} - 1)}$
30. $\dfrac{d}{dx}\dfrac{x^2 + 1}{x^2 + x + 1}$

Find the limits in Exercises 31–46.

31. $\lim_{x\to 1}(x^2 + 1)$
32. $\lim_{x\to 1}\dfrac{x^3 + 1}{x + 1}$
33. $\lim_{x\to 1}\dfrac{x^3 - 1}{x - 1}$
34. $\lim_{x\to 1}\dfrac{x^5 - 1}{x - 1}$
35. $\lim_{h\to 0}\dfrac{(h-2)^6 - 64}{h}$
36. $\lim_{h\to 0}\dfrac{(h-2)^6 + 64}{h}$
37. $\lim_{x\to 3}\dfrac{3x^2 + 2x}{x}$
38. $\lim_{x\to 0}\dfrac{3x^2 + 2x}{x}$
39. $\lim_{x\to 3}\dfrac{x^2 - 9}{x - 3}$
40. $\lim_{s\to 0}\dfrac{(s+3)^9 - 3^9}{s}$
41. $\lim_{\Delta x\to 0}\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ where $f(x) = x^4 + 3x^2 + 2$
42. $\lim_{\Delta x\to 0}\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ where $f(x) = 3x^8 - 8x^7 + 10$.
43. $\lim_{x\to\infty}\dfrac{3\sqrt{x} + 2}{5\sqrt{x} + 1}$
44. $\lim_{x\to\infty}\dfrac{5x^2 + 4}{3x^2 + 9}$
45. $\lim_{x\to\infty}\dfrac{5x^2 + 4}{5x^3 + 9}$
46. $\lim_{x\to\infty}\dfrac{5x^3 + 4}{5x^2 + 9}$
47. For the function in Fig. 1.R.1, find $\lim_{x\to x_0} f(x)$ for $x_0 = -3, -2, -1, 0, 1, 2, 3$. If the limit is not defined or does not exist, say so.

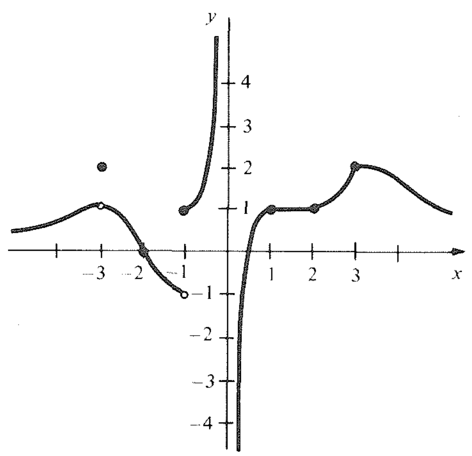


Figure 1.R.1. Find $\lim_{x\to x_0} f(x)$ at the indicated points.

48. Do as in Exercise 47 for the function in Fig. 1.R.2.

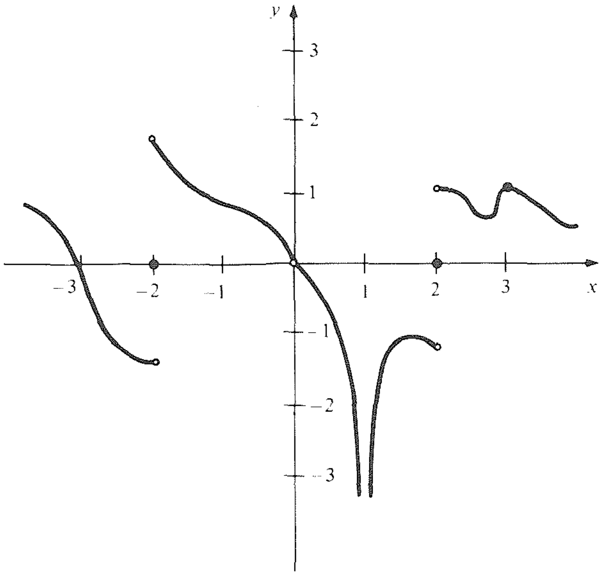


Figure 1.R.2. Find the limit at the indicated points.

49. Use the limit method directly to find $f'(1)$, where $f(x) = 3x^3 + 8x$.
50. Use the limit method directly to compute $(d/dx)(x^3 - 8x^2)$ at $x = -1$.
51. Use the limit method directly to compute $(d/dx)(x - \sqrt{x})$.
52. Use the limit method directly to compute the derivative of $f(x) = x^3 - \frac{1}{3}$.

Find the slope of the tangent line to the graphs of the functions in Exercises 53–58 at the indicated points.

53. $y = x^3 - 8x^2$; $x_0 = 1$
54. $y = x^4 + 2x$; $x_0 = -1$
55. $y = (x^2 + 1)(x^3 - 1)$; $x_0 = 0$
56. $y = x^4 + 10x + 2$; $x_0 = 2$
57. $y = 3x^4 - 10x^9$; $x_0 = 0$
58. $y = 3x + 1$; $x_0 = 5$
59. Two long trains, A and B, are moving on adjacent tracks with positions given by the functions $A(t) = t^3 + 2t$ and $B(t) = 7t^2/2 + 8$. What are the best times for a hobo on train B to make a moving transfer to train A?
60. A backpack is thrown down from a cliff at $t = 0$. It has fallen $2t + 4.9t^2$ meters after t seconds. Find its velocity at $t = 3$.
61. A bus moving along a straight road has moved $f(t) = (t^2 + \sqrt{t})/(1 + \sqrt{t})$ meters after time t (in seconds). What is its velocity at $t = 1$?
62. A car has position $x = (\sqrt{t} - 1)/(2\sqrt{t} + 1)$ at time t. What is its velocity at $t = 4$?

Calculate approximate values for the quantities in Exercises 63–70 using the linear approximation.

63. $(1.009)^8$
64. $(-1.008)^4 - 3(-1.008)^3 + 2$
65. $\sqrt{4.0001}$
66. $\sqrt{8.97}$
67. $f(2.003)$ where $f(x) = \dfrac{3x^3 - 10x^2 + 8x + 2}{x}$.
68. $g(1.0005)$ where $g(x) = x^4 - 10x^3$.
69. $h(2.95)$, where $h(s) = 4s^3 - s^4$.
70. $\dfrac{1 + (0.99)^2}{1 + (0.99)^3}$.

Find the equation of the line tangent to the graph of the function at the indicated point in Exercises 71–74.

71. $f(x) = x^3 - 6x$; $(0, 0)$
72. $f(x) = \dfrac{x^4 - 1}{6x^2 + 1}$; $(1, 0)$
73. $f(x) = \dfrac{x^3 - 7}{x^3 + 11}$; $(2, \frac{1}{19})$
74. $f(x) = \dfrac{x^5 - 6x^4 + 2x^3 - x}{x^2 + 1}$; $(1, -2)$
75. A sphere is increased in radius from 2 meters to 2.01 meters. Find the increase in volume using the linear approximation. Compare with the exact value.
76. A rope is stretched around the earth's equator. If it is to be raised 10 feet off the ground, approximately how much longer must it be? (The earth is 7,927 miles in diameter.)

In Exercises 77–80, let $f(x) = 2x^2 - 5x + 2$, $g(x) = \frac{3}{4}x^2 + 2x$ and $h(x) = -3x^2 + x + 3$.

77. Find the derivative of $f(x) + g(x)$ at $x = 1$.
78. Find the derivative of $3f(x) - 2h(x)$ at $x = 0$.
79. Find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.
80. Find the equation of the tangent line to the graph of $g(x)$ at $x = -1$.
81. Let B be a rectangular box with a square end of side length r. Suppose B is three times as long as it is wide. Let V be the volume of B. Compute dV/dr. What fraction of the surface area of B is your answer?
82. Calculate $\lim_{x\to\infty}(x - \sqrt{x^2 - a^2})$ and interpret your answer geometrically by drawing a right triangle with hypotenuse of length x and short leg of length a.
83. Suppose that $z = 2y^2 + 3y$ and $y = 5x + 1$.
    (a) Find dz/dy and dy/dx.
    (b) Express z in terms of x and find dz/dx.
    (c) Compare dz/dx with $(dz/dy) \cdot (dy/dx)$. (Write everything in terms of x.)
    (d) Solve for x in terms of y and find dx/dy.
    (e) Compare dx/dy with dy/dx.

84. Differentiate both sides of the equation

$$\frac{f(x)}{g(x)} = \frac{1}{g(x)/f(x)}$$

and show that you get the same result on each side.

★85. Find the equation of a line through the origin, with positive slope, which is tangent to the parabola $y = x^2 - 2x + 2$.

★86. Prove that the parabola $y = x^2$ has the optical focusing property mentioned in Section R.5. (This problem requires trigonometry; consult Section 5.1 for a review.) *Hint*: Refer to Fig. 1.R.3 and carry out the following program:

(a) Express $\tan\phi$ and $\tan\theta$ in terms of x.

(b) Prove that $90° - \theta = \theta - \phi$ by using the trigonometric identities:

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

and

$$\tan(\phi + 90°) = -\frac{1}{\tan\phi}.$$

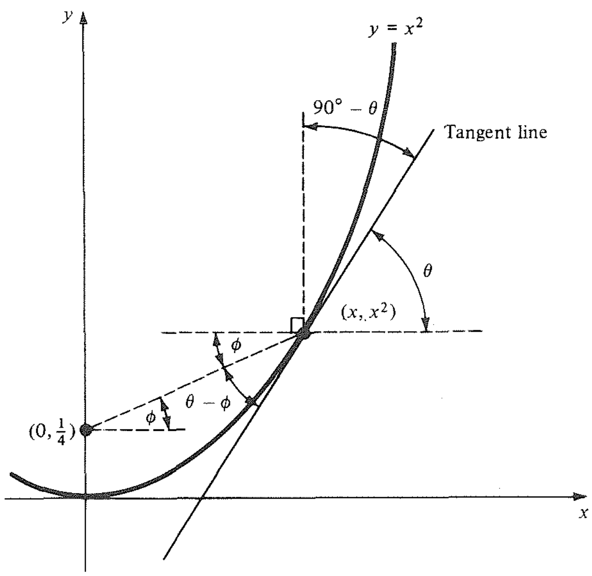


Figure 1.R.3. The geometry needed to prove that the parabola has the optical focusing property.

★87. Prove that the parabola $y = ax^2$ has the optical focusing property. (You should start by figuring out where the focal point will be.)

★88. The following is a useful technique for drawing the tangent line at a point P on a curve on paper (not given by a formula). Hold a mirror perpendicular to the paper and rotate it until the graph and its reflection together form a differentiable curve through P. Draw a line l along the edge of the mirror. Then the line through P perpendicular to l is the tangent line. (See Fig. 1.R.4.) Justify this procedure.

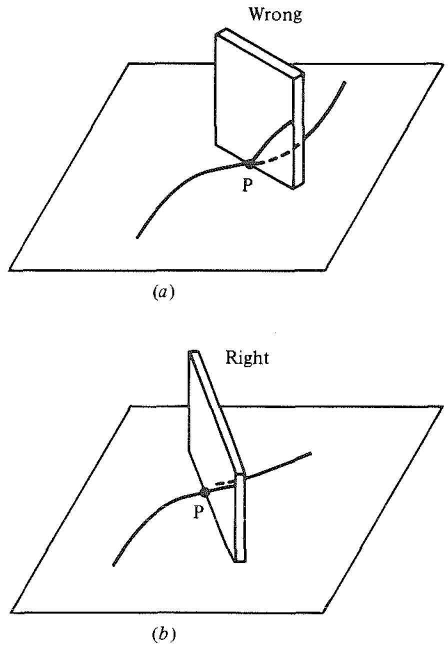


Figure 1.R.4. How to draw a tangent line with a mirror.

★89. The polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is said to have *degree* n if $a_n \neq 0$. For example: $\deg(x^3 - 2x + 3) = 3$, $\deg(x^4 + 5) = 4$, $\deg(0x^2 + 3x + 1) = 1$. The degree of the rational function $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials, is defined to be the degree of f minus the degree of g.

(a) Prove that, if $f(x)$ and $g(x)$ are polynomials, then $\deg f(x)g(x) = \deg f(x) + \deg g(x)$.

(b) Prove the result in part (a) when $f(x)$ and $g(x)$ are rational functions.

(c) Prove that, if $f(x)$ is a rational function with nonzero degree, then $\deg f'(x) = \deg f(x) - 1$. What if $\deg f(x) = 0$?

★90. Show that $f(x) = x$ and $g(x) = 1/(1 - x)$ obey the "false product rule" $(fg)'(x) \stackrel{?}{=} f'(x)g'(x)$.

★91. (a) Prove that if f/g is a rational function (i.e., a quotient of polynomials) with derivative zero, then f/g is a constant.

(b) Conclude that if the rational functions F and G are both antiderivatives for a function h, then F and G differ by a constant.