

Rates of Change and the Chain Rule

The rate at which one variable is changing with respect to another can be computed using differential calculus.

In Chapter 1, we learned how to differentiate algebraic functions and, thereby, to find velocities and slopes. In this chapter, we will learn some applications involving rates of change. We will also develop a new rule of differential calculus called the chain rule. This rule is important for our study of related rates in this chapter and will be indispensable when we come to use trigonometric and exponential functions.

2.1 Rates of Change and the Second Derivative

If $y = f(x)$, then $f'(x)$ is the rate of change of y with respect to x .

The derivative concept applies to more than just velocities and slopes. To explain these other applications of the derivative, we shall begin with the situation where two quantities are related linearly.

Suppose that two quantities x and y are related in such a way that a change Δx in x always produces a change Δy in y which is proportional to Δx ; that is, the ratio $\Delta y/\Delta x$ equals a constant, m . We say that y changes *proportionally* or *linearly* with x .

For instance, consider a hanging spring to which objects may be attached. Let x be the weight of the object in grams, and let y be the resulting length of the spring in centimeters. It is an experimental fact called Hooke's law that (for values of Δx which are not too large) a change Δx in the weight of the object produces a proportional change Δy in the length of the spring. (See Fig. 2.1.1.)

If we graph y against x , we get a segment of a straight line with slope

$$m = \frac{\Delta y}{\Delta x}$$

as shown in Fig. 2.1.2. The equation of the line is $y = mx + b$, and the

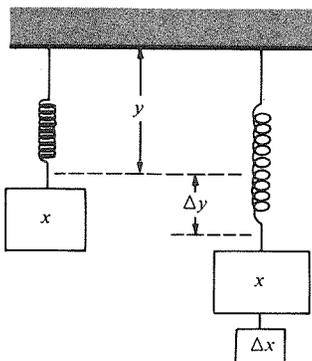


Figure 2.1.1. Hooke's law states that the change in length Δy is proportional to the change in weight Δx .

function $f(x) = mx + b$ is a *linear function*. The slope m of a straight line represents *the rate of change of y with respect to x* . (The quantity b is the length of the spring when the weight is removed.)

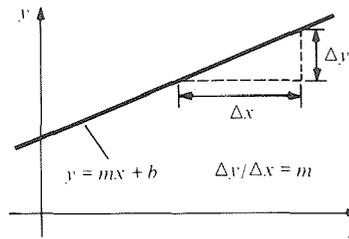


Figure 2.1.2. y changes proportionally to x when $\Delta y/\Delta x$ is constant.

Example 1 Suppose that y changes proportionally with x , and the rate of change is 3. If $y = 2$ when $x = 0$, find the equation relating y to x .

Solution The rate of change is the slope: $m = 3$. The equation of a straight line with this slope is $y = 3x + b$, where b is to be determined. Since $y = 2$ when $x = 0$, b must be 2; hence $y = 3x + 2$. ▲

Linear or Proportional Change

The variable y changes proportionally with x when y is related to x by a linear function: $y = mx + b$, where $\Delta y/\Delta x = m$. The number m is the rate of change of y with respect to x .

Example 2 Let S denote the supply of hogs in Chicago, measured in thousands, and let P denote the price of pork in cents per pound. Suppose that, for S between 0 and 100, P changes linearly with S . On April 1, $S = 50$ and $P = 163$; on April 3, a rise in S of 10 leads to a decline in P to 161. What happens if S falls to 30?

Solution Watch the words *of* and *to*! The rise in S of 10 means that $\Delta S = 10$; the decline of P to 161 means that $\Delta P = 161 - 163 = -2$. Thus the rate of change is $-\frac{2}{10} = -\frac{1}{5}$. (The minus sign indicates that the direction of price change is opposite to the direction of supply change.) We have $P = -\frac{1}{5}S + b$ for some b . Since $P = 163$ when $S = 50$, we have $163 = -\frac{1}{5} \cdot 50 + b$, or $b = 163 + 10 = 173$, so $P = -\frac{1}{5}S + 173$. When $S = 30$, this gives $P = -6 + 173 = 167$. At this point, then, pork will cost \$1.67 a pound. ▲

If the dependence of $y = f(x)$ on x is not linear, we can still introduce the notion of the *average rate of change* of y with respect to x , just as we introduced the average velocity in Section 1.1. Namely, the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called *the average rate of change of y with respect to x on the interval between x_0 and $x_0 + \Delta x$* . For functions f which are not linear, this average rate of change depends on the interval chosen. If we fix x_0 and let Δx approach 0, the limit of the average rate of change is the derivative $f'(x_0)$, which we refer to as *the rate of change of y with respect to x at the point x_0* . This may be referred to as an *instantaneous* rate of change, especially when the independent variable represents time.

Example 3 An oil slick has area $y = 30x^3 + 100x$ square meters x minutes after a tanker explosion. Find the average rate of change in area with respect to time during the period from $x = 2$ to $x = 3$ and from $x = 2$ to $x = 2.1$. What is the instantaneous rate of change of area with respect to time at $x = 2$?

Solution The average rate of change from $x = 2$ to $x = 3$ is

$$\frac{(30 \cdot 3^3 + 100 \cdot 3 - 30 \cdot 2^3 - 100 \cdot 2) \text{ square meters}}{1 \text{ minute}} = 670 \frac{\text{square meters}}{\text{minute}}$$

From $x = 2$ to $x = 2.1$, the average rate is calculated in a similar way to be

$$\frac{47.83}{0.1} = 478.3 \frac{\text{square meters}}{\text{minute}}$$

Finally, the instantaneous rate of change is found by evaluating the derivative $90x^2 + 100$ at $x = 2$ to obtain 460. Since the instantaneous rate of change is a limit of average rates, it is measured in the same units, so the oil slick is growing at a rate of 460 square meters per minute after 2 minutes. ▲

Rates of Change

If two quantities x and y are related by $y = f(x)$, the derivative $f'(x_0)$ represents the rate of change of y with respect to x at the point x_0 . It is measured in (units of y)/(units of x).

A positive rate of change is sometimes called a *rate of increase*.

Example 4 A circle with radius r millimeters has area $A = \pi r^2$ square millimeters. Find the rate of increase of area with respect to radius at $r_0 = 5$. Interpret your answer geometrically.

Solution Here $A = f(r) = \pi r^2$. Since π is a constant, the derivative $f'(r)$ is $2\pi r$, and $f'(5) = 10\pi$. Notice that the rate of change is measured in units of (square millimeters)/millimeters, which are just millimeters. The value $2\pi r$ of the rate of change can be interpreted as the circumference of the circle (Fig. 2.1.3). ▲

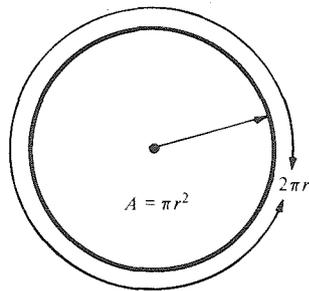


Figure 2.1.3. The rate of change of A with respect to r is $2\pi r$, the circumference of the circle.

In the next two examples, a negative rate of change indicates that one quantity decreases when another increases. Since $\Delta y = f(x_0 + \Delta x) - f(x_0)$, it follows that Δy is negative when $f(x_0 + \Delta x) < f(x_0)$. Thus, if $\Delta y/\Delta x$ is negative, an increase in x produces a decrease in y . This leads to our stated interpretation of negative rates of change. If a rate of change is negative, its absolute value is sometimes called a *rate of decrease*.

Example 5 Suppose that the price of pork P depends on the supply S by the formula $P = 160 - 3S + (0.01)S^2$. Find the rate of change of P with respect to S when $S = 50$. (See Example 2 for units.)

Solution The rate of change is the derivative of $f(S) = 0.01S^2 - 3S + 160$ with respect to $x = S$ at $x_0 = 50$. The derivative is $f'(S) = 0.02S - 3$. When $S = 50$, we get $f'(50) = 1 - 3 = -2$. Thus the price is *decreasing* by 2 cents per pound per thousand hogs when $S = 50$ thousand. ▲

Example 6 A reservoir contains $10^8 - 10^4t - 80t^2 - 10t^3 + 5t^5$ liters of water at time t , where t is the time in hours from when the gates are opened. How many liters per hour are leaving the reservoir after one hour?

Solution The rate of change of the amount of water in the reservoir is the derivative of the polynomial $10^8 - 10^4t - 80t^2 - 10t^3 + 5t^5$, namely $-10^4 - 160t - 30t^2 + 25t^4$. At $t = 1$, this equals $-10^4 - 160 - 30 + 25 = -10,165$ liters per hour. This is negative, so 10,165 liters per hour are *leaving* the reservoir after one hour. ▲

We now reconsider the velocity and acceleration of a particle moving on a straight line. Suppose for the moment that the line is vertical and designate one direction as “+” and the other as “-”. We shall usually choose the upward direction as “+,” but consistently using the other sign would give equivalent results. We also choose some point as the origin, designated by $x = 0$, as well as a unit of length, such as meters. Thus, if our designated origin represents the level of the Golden Gate Bridge, $x = 100$ would designate a location 100 meters above the bridge along our vertical straight line, and $x = -10$ would indicate a location 10 meters beneath the bridge (Fig. 2.1.4).

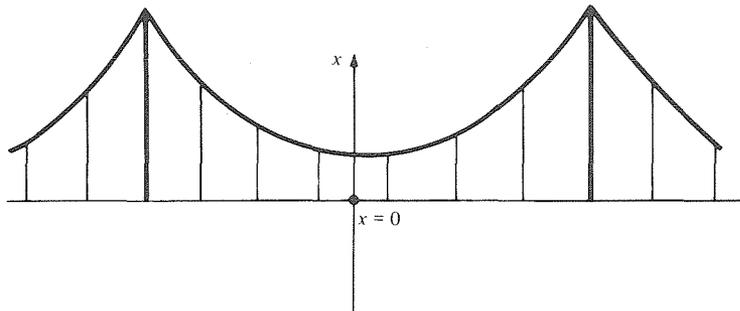


Figure 2.1.4. A coordinate system with “+” upwards and $x = 0$ at bridge level.

Suppose that, at time t , a particle has location $x = f(t)$ along our line. We call $[f(t + \Delta t) - f(t)]/\Delta t$ the average velocity and $dx/dt = f'(t)$ the instantaneous velocity; this can either be positive, indicating upward motion, or can be negative, indicating downward motion.

Example 7 Suppose that $x = 0$ represents the level of the Golden Gate Bridge and that $x = f(t) = 8 + 6t - 5t^2$ represents the position of a stone at time t in seconds.

- Is the stone above the bridge, at the bridge, or below the bridge at $t = 0$? How about at $t = 2$?
- Suppose that the average velocity during the interval from t_0 to $t_0 + \Delta t$ is negative; what can be said about the height at time $t_0 + \Delta t$?
- What is the instantaneous velocity at $t = 1$?

Solution (a) At $t = 0$, $x = 8$, so the stone is 8 meters above the level of the bridge. At $t = 2$, $x = 8 + 6 \cdot 2 - 5 \cdot 4 = 0$, so the stone is at the level of the bridge.
 (b) It is less than that at time t_0 .
 (c) We compute $dx/dt = 6 - 10t$, which at $t = 1$ is -4 . Thus, the instantaneous velocity is 4 meters per second downward. ▲

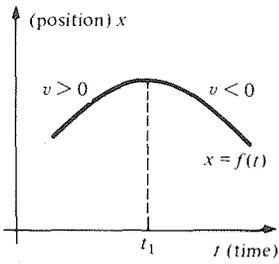


Figure 2.1.5. The velocity is positive until $t = t_1$ and negative afterward.

Of course, these interpretations of positive and negative velocity also apply to horizontal motion, like that which we discussed at the beginning of Chapter 1. In particular, we may let x denote the position along a road with larger values of x corresponding to points further east, say. Then $dx/dt > 0$ indicates that motion is eastward; $dx/dt < 0$ indicates westward motion. The magnitude, or absolute value, of the velocity is called the *speed*.

If we graph $x = f(t)$ against t , the slope of the graph indicates whether the velocity is positive or negative. (See Fig. 2.1.5.)

Note that the instantaneous velocity $v = dx/dt = f'(t)$ is usually itself changing with time. The rate of change of v with respect to time is called *acceleration*; it may be computed by differentiating $v = f'(t)$ once again.

Example 8 Suppose that $x = f(t) = \frac{1}{4}t^2 - t + 2$ denotes the position of a bus at time t .

- (a) Find the velocity as a function of time; plot its graph.
- (b) Find and plot the speed as a function of time.
- (c) Find the acceleration.

Solution (a) The velocity is $v = dx/dt = \frac{1}{2}t - 1$ (see Fig. 2.1.6(a)).

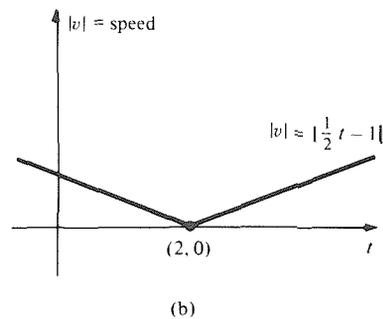
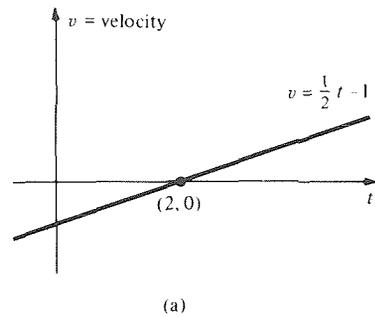


Figure 2.1.6. (a) Velocity and (b) speed as functions of time for $x = \frac{1}{4}t^2 - t + 2$. The acceleration dv/dt is constant and positive.

(b) The speed $|v| = |\frac{1}{2}t - 1|$ [see Fig. 2.1.6(b)].

(c) The acceleration is $dv/dt = \frac{1}{2}$. ▲

In this example, the acceleration happens to be constant and positive, indicating that the *velocity* is increasing at a constant rate. Note, though, that the *speed* decreases and then increases; it decreases when the velocity and acceleration have opposite signs and increases when the signs are the same. This may be illustrated by an example. If your car is moving backwards (negative velocity) but you have a positive acceleration, your speed decreases until your car reverses direction, moves forward (positive velocity), and the speed increases.

Since acceleration is the derivative of the velocity and velocity is already

a derivative, we have an example of the general concept of the *second derivative*, i.e., the derivative of the derivative. If $y = f(x)$, the second derivative is denoted $f''(x)$ and is defined to be the derivative of $f'(x)$. In Leibniz notation we write d^2y/dx^2 for the second derivative of $y = f(x)$. Note that d^2y/dx^2 is *not* the square of dy/dx , but rather represents the result of the operation d/dx performed twice.

If an object has position x (in meters) which is a function $x = f(t)$ of time t (in seconds), the acceleration is thus denoted by $f''(t)$ or d^2x/dt^2 . It is measured in meters per second per second, i.e., meters per second² or feet per second².

Second Derivatives

To compute the second derivative $f''(x)$:

1. Compute the first derivative $f'(x)$.
2. Calculate the derivative of $f'(x)$; the result is $f''(x)$.

The second derivative of $y = f(x)$ is written in Leibniz notation as

$$\frac{d^2y}{dx^2}.$$

The second derivative of position with respect to time is called acceleration.

If we plot the graph $y = f(x)$, we know that $f'(x)$ represents the slope of the tangent line. Thus, if the second derivative is positive, the slope must be increasing as we move to the right, as in Fig. 2.1.7(a). Likewise, a negative second derivative means that the slope is decreasing as we increase x , as in Fig. 2.1.7(b).

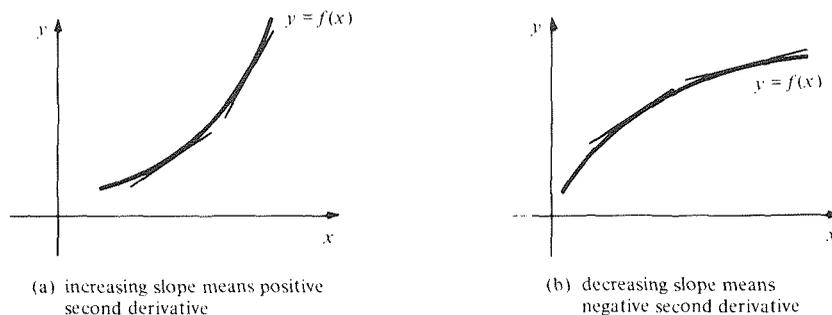


Figure 2.1.7. The rate of change of slope with respect to x is the second derivative $f''(x)$. (a) $f'' > 0$. (b) $f'' < 0$.

Example 9 Calculate the second derivative of

(a) $f(x) = x^4 + 2x^3 - 8x$, (b) $f(x) = \frac{x+1}{\sqrt{x}}$.

(c) $\frac{d^2}{dx^2}(3x^2 - 2x + 1)$, (d) $\frac{d^2}{dr^2}(8r^2 + 2r + 10)$.

Solution (a) By our rules for differentiating polynomials from Section 1.4,

$$f'(x) = 4x^3 + 6x^2 - 8.$$

Now we differentiate this new polynomial:

$$f''(x) = 12x^2 + 12x - 0 = 12(x^2 + x).$$

(b) By the quotient rule from Section 1.5,

$$f'(x) = \frac{\sqrt{x} - (x+1) \cdot 1/2\sqrt{x}}{x} = \frac{x-1}{2\sqrt{x} \cdot x} = \frac{x-1}{2x^{3/2}}.$$

By the quotient rule again,

$$f''(x) = \frac{1}{2} \left(\frac{x^{3/2} - (x-1) \cdot \frac{3}{2}x^{1/2}}{x^3} \right) = \frac{3\sqrt{x} - \sqrt{x^3}}{4x^3}.$$

(c) Here, $\frac{d(3x^2 - 2x + 1)}{dx} = 6x - 2$,

$$\text{and so } \frac{d^2(3x^2 - 2x + 1)}{dx^2} = \frac{d}{dx}(6x - 2) = 6,$$

a constant function.

$$\begin{aligned} \text{(d) } \frac{d^2}{dr^2}(8r^2 + 2r + 10) &= \left(\frac{d}{dr} \right) \left[\left(\frac{d}{dr} \right) (8r^2 + 2r + 10) \right] \\ &= \left(\frac{d}{dr} \right) [16r + 2] = 16. \blacktriangle \end{aligned}$$

Next we consider a word problem involving second derivatives.

Example 10 A race car travels $\frac{1}{4}$ mile in 6 seconds, its distance from the start in feet after t seconds being $f(t) = 44t^2/3 + 132t$.

- (a) Find its velocity and acceleration as it crosses the finish line.
 (b) How fast was it going halfway down the track?

Solution (a) The velocity at time t is $v = f'(t) = 88t/3 + 132$, and the acceleration is $a = f''(t) = \frac{88}{3}$. Substituting $t = 6$, we get $v = 308$ feet per second (= 210 miles per hour) and $a \approx 29.3$ feet per second².
 (b) To find the velocity halfway down, we do *not* substitute $t = 3.00$ in $v = f'(t)$ —that would be its velocity after half the time has elapsed. The total distance covered is $f(6) = (44)(36)/3 + (132)(6) = 1320$ feet (= $\frac{1}{4}$ mile). Thus, half the distance is 660 feet. To find the time t corresponding to the distance 660, we write $f(t) = 660$ and solve for t using the quadratic formula:

$$\frac{44t^2}{3} + 132t = 660,$$

$$t^2 + 9t - 45 = 0 \quad (\text{multiply by } \frac{3}{44}),$$

$$t = \frac{-9 \pm \sqrt{81 + 180}}{2} \approx -12.58, 3.58 \quad (\text{quadratic formula}).$$

Since the time during the race is positive, we discard the negative root and retain $t = 3.58$. Substituting into $v = f'(t) = 88t/3 + 132$ gives $v \approx 237$ feet per second (≈ 162 miles per hour). \blacktriangle

We end this section with a discussion of some concepts from economics, where special names are given to certain rates of change.

Imagine a factory in which x worker-hours of labor can produce $y = f(x)$ dollars worth of output. First, suppose that y changes proportionally with x .

Then $\Delta y = f(x_0 + \Delta x) - f(x_0)$ represents the amount of extra output produced if Δx extra worker-hours of labor are employed. Thus, $\Delta y/\Delta x$ is the output per worker-hour. This average rate of change is called the *productivity of labor*.

Next, suppose that $f(x)$ is not necessarily linear. Then $\Delta y/\Delta x$ is the extra output per extra worker-hour of extra labor when Δx extra worker-hours are employed. The limiting value, as Δx becomes very small, is $f'(x_0)$. This instantaneous rate of change is called the *marginal productivity of labor* at the level x_0 .

In Fig. 2.1.8 we sketch a possible productivity curve $y = f(x)$. Notice that as x_0 becomes larger and larger, the marginal productivity $f'(x_0)$ (= dollars of output per worker-hour at level x_0) becomes smaller. One says that the *law of diminishing returns* applies.

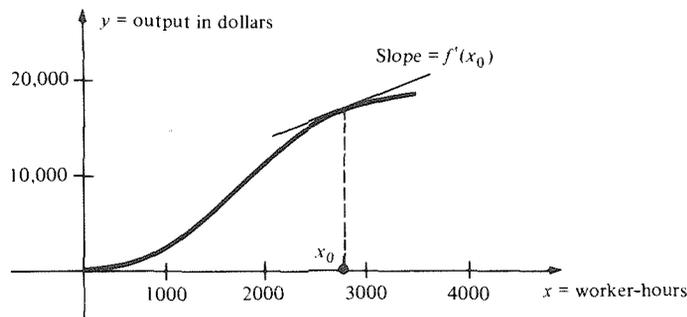


Figure 2.1.8. A possible productivity curve; the slope of the tangent line is the marginal productivity.

Example 11 A bagel factory produces $30x - 2x^2 - 2$ dollars worth of bagels for each x worker hours of labor. Find the marginal productivity when 5 worker hours are employed.

Solution The output is $f(x) = 30x - 2x^2 - 2$ dollars. The marginal productivity at $x_0 = 5$ is $f'(5) = 30 - 4 \cdot 5 = 10$ dollars per worker-hour. Thus, at $x_0 = 5$, production would increase by 10 dollars per additional worker-hour. ▲

Next we discuss marginal cost and marginal revenue. Suppose that a company makes x calculators per week and that the management is free to adjust x . Define the following quantities:

- $C(x)$ = the *cost* of making x calculators (labor, supplies, etc.)
- $R(x)$ = the *revenue* obtained by producing x calculators (sales).
- $P(x) = R(x) - C(x)$ = the *profit*.

Even though $C(x)$, $R(x)$, and $P(x)$ are defined only for integers x , economists find it useful to imagine them defined for all real x . This works nicely if x is so large that a change of one unit, $\Delta x = 1$, can legitimately be called “very small.”

The derivative $C'(x)$ is called the *marginal cost* and $R'(x)$ is the *marginal revenue*:

$$C'(x) = \text{marginal cost} = \left\{ \begin{array}{l} \text{the cost per calculator for producing} \\ \text{additional calculators at production level } x. \end{array} \right.$$

$$R'(x) = \text{marginal revenue} = \left\{ \begin{array}{l} \text{the revenue per calculator obtained by} \\ \text{producing additional calculators at} \\ \text{production level } x. \end{array} \right.$$

Since $P(x) = R(x) - C(x)$, we get $P'(x) = R'(x) - C'(x)$, the profit per additional calculator at production level x . This is the *marginal profit*. If the price

per unit is $f(x)$ and x calculators are sold, then $R(x) = xf(x)$. By the product rule, the marginal revenue is $R'(x) = xf'(x) + f(x)$.

Example 12 Suppose that it costs $(30x + 0.04x^2)/(1 + 0.0003x^3)$ dollars if x calculators are made, where $0 \leq x \leq 100$, and that calculators are priced at $100 - 0.05x$ dollars. If all x calculators are sold, what is the marginal profit?

Solution The revenue is $R = x(100 - 0.05x)$, so the profit is $P = x(100 - 0.05x) - (30x + 0.04x^2)/(1 + 0.0003x^3)$. The marginal profit is therefore dP/dx , which may be calculated using the sum and quotient rules as follows:

$$\begin{aligned} \frac{dP}{dx} &= \frac{d}{dx}(100x - 0.05x^2) - \frac{d}{dx} \left(\frac{30x + 0.04x^2}{1 + 0.0003x^3} \right) \\ &= 100 - 0.1x - \frac{(30 + 0.08x)(1 + 0.0003x^3) - (30x + 0.04x^2)(0.0009x^2)}{(1 + 0.0003x^3)^2} \\ &= 100 - 0.1x - \frac{30 + 0.08x - .018x^3 - 0.000012x^4}{(1 + 0.0003x^3)^2} \cdot \blacktriangle \end{aligned}$$

Exercises for Section 2.1

In Exercises 1–4, assume that y changes proportionally with x and the rate of change is r . In each case, find y as a function of x , as in Example 1.

- $r = 5$, $y = 1$ when $x = 4$.
- $r = -2$, $y = 10$ when $x = 15$.
- $r = \frac{1}{2}$, $y = 1$ when $x = 3$.
- $r = 10$, $y = 4$ when $x = -1$.
- If the price of electricity changes proportionally with time, and if the price goes from 2 cents per kilowatt-hour in 1982 to 3.2 cents per kilowatt-hour in 1984, what is the rate of change of price with respect to time? When will the price be 5 cents per kilowatt-hour? What will the price be in 1991?
- It will take a certain woman seven bags of cement to build a 6-meter-long sidewalk of uniform width and thickness. Her husband offers to contribute enough of his own labor to extend the sidewalk to 7 meters. How much more cement do they need?
- A rock is thrown straight down the face of a vertical cliff with an initial velocity of 3 meters per second. Two seconds later, the rock is falling at a velocity of 22.6 meters per second. Assuming that the velocity v changes proportionally with time t , find the equation relating v to t . How fast is the rock falling after 15 seconds?
- In November 1980, Mr. B used 302 kilowatt-hours of electricity and paid \$18.10 to do so. In December 1980, he paid \$21.30 for 366 kilowatt-hours. Assuming that the cost of electricity changes linearly with the amount used, how much would Mr. B pay if he used no electricity at all? Suppose that Mr. B can reduce his bill to zero by selling solar-generated electricity back to

the company. How much must he sell? Interpret your answers on a graph.

Find the average rate of change of the functions in Exercises 9–12 on the specified interval.

- $f(t) = 400 - 20t - 16t^2$; t between $t_0 = 1$ and $t_1 = \frac{3}{2}$.
- $g(t) = 18t^2 + 2t + 3$; t between $t_0 = 2$ and $t_1 = 3.5$.
- $f(x) = (x + \frac{1}{2})^2$; $x_0 = 2$, $\Delta x = 0.5$.
- $g(s) = (3s + 2)(s - 1) - 3s^2$; $s_0 = 0$, $\Delta s = 6$.
- The volume of a cone is $\frac{1}{3}(\text{area of base}) \times \text{height}$. If the base has radius always equal to the height, find the rate of change of the volume with respect to this radius.
- Find the rate of change of the area of an equilateral triangle with respect to the length of one of its sides.
- During takeoff, a 747 has $25,000 - 80t + 2t^2 + 0.2t^3$ gallons of fuel in its tanks t seconds after starting its takeoff, $0 \leq t \leq 10$. How many gallons per second are being burned 2 seconds into the takeoff?
- A space shuttle's external tank contains $10^5 - 10^4t - 10^3t^3$ liters of fuel t minutes after blastoff. How many liters per minute are being burned two minutes after blastoff?
- If the height H in feet of a certain species of tree depends on its base diameter d in feet through the formula $H = 56d - 3d^2$, find the rate of change of H with respect to d at $d = 0.5$.
- Suppose that tension T of a muscle is related to the time t of exertion by $T = 5 + 3t - t^2$, $0 \leq t \leq \frac{3}{2}$. Find the rate of change of T with respect to t at $t = 1$.

19. A flu epidemic has infected $P = 30t^2 + 100t$ people by t days after its outbreak. How fast is the epidemic spreading (in people per day) after 5 days?
20. Find the rate of change of the area of a circle with respect to its diameter when the diameter is 10. Compare with Example 4.
21. A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$. What is the rate of change of the volume of the sphere with respect to its radius? Give a geometric interpretation of the answer.
22. A balloon being blown up has a volume $V = 3t^3 + 8t^2 + 16t$ cubic centimeters after t minutes. What is the rate of change of volume (in cubic centimeters per minute) at $t = 0.5$?
23. Let $x(t) = t^3 + ct$ be the position of a particle at time t . For which values of c does the particle reverse direction, and at what times does the reversal take place for each such value of c ? Does the value of c affect the particle's acceleration?
24. An evasive moth has position $t^3 - t + 2$ at time t . A hungry bat has position $y(t) = -\frac{2}{3}t^2 + t + 2$ at time t . How many chances does the bat have to catch the moth? How fast are they going and what are their accelerations at these times?
25. If the position of a moving object at time t is $(t^5 + 1)(t + 2)$, find its velocity and acceleration when $t = 0.1$.
26. Let $h(t) = 2t^3$ be the position of an object moving along a straight line at time t . What are the velocity and acceleration at $t = 3$?

Compute the second derivatives in Exercises 27–32.

27. $\frac{d^2}{dx^2}(x^4 - 3x^2)$ 28. $\frac{d^2}{dx^2}(3x^2 - 8x + 10)$
29. $\frac{d^2}{dx^2}\frac{x^2 + 1}{x + 2}$ 30. $\frac{d^2}{dx^2}\frac{x^3 - 1}{x^4 + 8}$
31. $\frac{d^2}{dr^2}(3r^8 - 8r^7 + 10)$ 32. $\frac{d^2}{ds^2}(s^9 - 10s^8 + 5)$

Find the second derivative of the functions in Exercises 33–40.

33. $f(x) = x^2 - 5$
34. $f(x) = x - 2$
35. $y = x^5 + 7x^4 - 2x + 3$
36. $y = [(x - 1) + x^2][x^3 - 1]$
37. $y = \frac{x^2}{x - 1}$
38. $y = x^5 + \frac{1}{x} + \frac{2}{x^3}$
39. $\frac{t^2 + 1}{t^2 - 1}$
40. $\frac{s}{s + 1}$

In Exercises 41–46, find the velocities and accelerations at the indicated times of the particles whose positions y (in meters) on a line are given by the following func-

tions of time t (in seconds):

41. $y = 3t + 2$; $t_0 = 1$
42. $y = 5t - 1$; $t_0 = 0$
43. $y = 8t^2 + 1$; $t_0 = 0$
44. $y = 18t^2 - 2t + 5$; $t_0 = 2$
45. $y = 10 - 2t - 0.01t^4$; $t_0 = 0$
46. $y = 20 - 8t - 0.02t^6$; $t_0 = 1$

47. The height of a pebble dropped off a building at time $t = 0$ is $h(t) = 44.1 - 4.9t^2$ meters at time t . The pebble strikes the ground at $t = 3.00$ seconds.
- (a) What is its velocity and acceleration when it strikes the ground?
- (b) What is its velocity when it is halfway down the building?
48. The amount of rain y in inches at time x in hours from the start of the September 3, 1975 Owens Valley thunderstorm was given by $y = 2x - x^2$, $0 \leq x \leq 1$.
- (a) Find how many inches of rain per hour were falling halfway through the storm.
- (b) Find how many inches of rain per hour were falling after half an inch of rain has fallen.
49. A shoe repair shop can produce $20x - x^2 - 3$ dollars of revenue every hour when x workers are employed. Find the marginal productivity when 5 workers are employed.
50. The owners of a restaurant find that they can serve $300w - 2w^2 - 14$ dinners when w worker-weeks of labor are employed. If an average dinner is worth \$7.50, what is the marginal productivity (in dollars) of a worker when 10 workers are employed?
51. A factory employing w workers produces $100w + w^2/100 - (1/5000)w^4$ dollars worth of tools per day. Find the marginal productivity of labor when $w = 20$.
52. A farm can grow $10000x - 35x^3$ dollars worth of tomatoes if x tons of fertilizer are used. Find the marginal productivity of the fertilizer when $x = 10$. Interpret the sign of your answer.
53. In a boot factory, the cost in dollars of making x boots is $(4x + 0.02x^2)/(1 + 0.002x^3)$. If boots are priced at $25 - 0.02x$ dollars, what is the marginal profit, assuming that x boots are sold?
54. In a pizza parlor, the cost in dollars of making x pizzas is $(5x + 0.01x^2)/(1 + 0.001x^3)$. The price per pizza sold is set by the rule: price = $7 - 0.05x$ if x pizzas are made. If all x pizzas are sold, what is the marginal profit?
- In each of Exercises 55–58, what name would you give to the rate of change of y with respect to x ? In what units could this rate be expressed?
55. x = amount of fuel used; y = distance driven in an automobile.
56. x = distance driven in an automobile; y = amount of fuel used.

57. x = amount of fuel purchased; y = amount of money paid for fuel.
58. x = distance driven in an automobile; y = amount of money paid for fuel.
59. The cost c of fuel for driving, measured in cents per kilometer, can be written as the product $c = rp$, where r is the fuel consumption rate in liters per kilometer and p is the price of fuel in cents per liter. If r and p depend on time (the car deteriorates, price fluctuates), so does c . The rates of change are connected by the product rule

$$\frac{dc}{dt} = r \frac{dp}{dt} + \frac{dr}{dt} p.$$

Interpret in words each of the terms on the right-hand side of this equation, and explain why dc/dt should be their sum.

60. If $f(x)$ represents the cost of living at time x , then $f'(x) > 0$ means that there is inflation.
- What does $f''(x) > 0$ mean?
 - A government spokesman says, "The rate at which inflation is getting worse is decreasing." Interpret this statement in terms of $f'(x)$, $f''(x)$, and $f'''(x)$.
61. Let $y = 4x^2 - 2x + 7$. Compute the average rate of change of y with respect to x over the interval from $x_0 = 0$ to $x_1 = \Delta x$ for the following values of Δx : 0.1, 0.001, 0.000001. Compare with the derivative at $x_0 = 0$.
62. Repeat Exercise 61 with $\Delta x = -0.1$, -0.001 , and -0.000001 .
63. Find the average rate of change of the following functions on the given interval. Compare with the derivative at the midpoint.
- $f(x) = (x - \frac{1}{2})(x + 1)$ between $x = -\frac{1}{2}$ and $x = 0$.
 - $g(t) = 3(t + 5)(t - 3)$ on $[2, 6]$.
 - $h(r) = 10r^2 - 3r + 6$ on $[-0.1, 0.4]$.
 - $r(t) = (2 - t)(t + 4)$; t in $[3, 7]$.
64. (a) Let $y = ax^2 + bx + c$, where a , b , and c are constant. Show that the average rate of change of y with respect to x on any interval $[x_1, x_2]$ equals the instantaneous rate of change at the midpoint; i.e., at $(x_1 + x_2)/2$.
- (b) Let $f(x) = ax^2 + bx + c$, where a , b , and c are constant. Prove that, for any x_0 ,
- $$f(x) = f(x_0) + f'(m)(x - x_0)$$
- where $m = (x + x_0)/2$.
65. The length l and width w of a rectangle are functions of time given by $l = (3 + t^2 + t^3)$ centimeters and $w = (5 - t + 2t^2)$ centimeters at time t (in seconds). What is the rate of change of area with respect to time at time t ?
66. If the height and radius of a right circular cylinder are functions of time given by $h = (1 + t^2 + t^3 + t^4)$ centimeters and $r = (1 + 2t - t^2 + t^5)$ centimeters at time t (in seconds), what is the rate of change with respect to time of the lateral surface area, i.e., the total surface area minus the top and bottom?
67. Let $f(t) = 2t^2 - 5t + 2$ be the position of object A and let $h(t) = -3t^2 + t + 3$ be the position of object B .
- When is A moving faster than B ?
 - How fast is B going when A stops?
 - When does B change direction?
68. Repeat Example 7 with the same data but with the following two conventions changed. First, the origin is now chosen at a point 20 meters above the bridge. Second, we designate down as "+" and up as "-" rather than vice versa.
69. For which functions $f(x) = ax^2 + bx + c$ is the second derivative equal to the zero function?
70. How do the graphs of functions $ax^2 + bx + c$ whose second derivative is positive compare with those for which the second derivative is negative and those for which the second derivative is zero?
71. A particle is said to be *accelerating* (or *decelerating*) if the sign of its acceleration is the same as (or opposite to) the sign of its velocity. (a) Let $f(t) = -t^3$ be the position of a particle on a straight line at time t . When is the particle accelerating and when is it decelerating? (b) If the position of a particle on a line is given as a quadratic function of time and the particle is accelerating at time t_0 , does the particle ever decelerate?
72. One summer day in Los Angeles, the pollution index at 7:00 AM was 20 parts per million, increasing linearly 15 parts per million each hour until 5:00 PM. Let y be the amount of pollutants in the air x hours after 7:00 AM.
- Find a linear equation relating y and x .
 - The slope is the increase in pollution for each hour increase in time. Find it.
 - Find the pollution level at 5:00 PM.
73. Straight-line depreciation means that the difference between current value and original value is directly proportional to the time t . Suppose a home office is presently furnished for \$4000 and salvaged for \$500 after ten years. Assume straight line depreciation.
- Find a linear equation for the value V of the office furniture after t years, for tax purposes.
 - The slope of the line indicates the decrease in value each year of the office furniture, to be used in preparing a tax return. Find it.

- ★74. Suppose that the acceleration of an object is constant and equal to 9.8 meters/second² and that its velocity at time $x = 0$ is 2 meters/second.
- Express the velocity as a function of x .
 - What is the velocity when $x = 3$?
 - Express the position y of the object as a function of x , if $y = 4$ when $x = 0$.
 - How far does the object travel between $x = 2$ and $x = 5$?

★75. Let

$$f(x) = \begin{cases} 0, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

- Sketch a graph of $f(x)$.
- Find $f'(x)$. Sketch its graph.
- Find $f''(x)$ for $x \neq 0$. Sketch its graph. What happens when $x = 0$?
- Suppose that $f(x)$ is the position of an object at time x . What might have happened at $x = 0$?

2.2 The Chain Rule

The derivative of $f(g(x))$ is a product of derivatives.

None of the rules which we have derived so far tell us how to differentiate $\sqrt{x^3 - 5} = (x^3 - 5)^{1/2}$. The chain rule will. Before deriving this rule, though, we shall look at what happens when we differentiate a function raised to an integer power.

If $g(x)$ is any function, we can use the product rule to differentiate $[g(x)]^2$:

$$\frac{d}{dx} [g(x)]^2 = \frac{d}{dx} [g(x)g(x)] = g'(x)g(x) + g(x)g'(x) = 2g(x)g'(x).$$

If we write $u = g(x)$, this can be expressed in Leibniz notation as

$$\frac{d}{dx} (u^2) = 2u \frac{du}{dx}.$$

In the same way, we may differentiate u^3 :

$$\begin{aligned} \frac{d}{dx} (u^3) &= \frac{d}{dx} (u^2 \cdot u) = \frac{d}{dx} (u^2) \cdot u + u^2 \frac{du}{dx} \\ &= 2u \frac{du}{dx} \cdot u + u^2 \frac{du}{dx} = 3u^2 \frac{du}{dx}. \end{aligned}$$

Similarly, $(d/dx)(u^4) = 4u^3(du/dx)$ (check it yourself); and, for a general positive integer n , we have $(d/dx)u^n = nu^{n-1}(du/dx)$. (This may be formally proved by induction—see Exercise 52.)

Power of a Function Rule

To differentiate the n th power $[g(x)]^n$ of a function $g(x)$, where n is a positive integer, take out the exponent as a factor, reduce the exponent by 1, and multiply by the derivative of $g(x)$:

$$(g^n)'(x) = n[g(x)]^{n-1}g'(x),$$

$$\frac{d}{dx} (u^n) = nu^{n-1} \frac{du}{dx}$$

If $u = x$, then $du/dx = 1$, and the power of a function rule reduces to the ordinary power rule.

A common mistake made by students in applying the power of a function rule is to forget the extra factor of $g'(x)$ —that is, du/dx .

Example 1 Find the derivative of $[g(x)]^3$, where $g(x) = x^4 + 2x^2$, first by using the power of a function rule and then by expanding the cube and differentiating directly. Compare the answers.

Solution By the power of a function rule, with $u = x^4 + 2x^2$ and $n = 3$,

$$\frac{d}{dx}(x^4 + 2x^2)^3 = 3(x^4 + 2x^2)^2 \frac{d}{dx}(x^4 + 2x^2) = 3(x^4 + 2x^2)^2(4x^3 + 4x).$$

If we expand the cube first, we get $(x^4 + 2x^2)^3 = x^{12} + 6x^{10} + 12x^8 + 8x^6$, so

$$\begin{aligned} \frac{d}{dx}(x^4 + 2x^2)^3 &= \frac{d}{dx}(x^{12} + 6x^{10} + 12x^8 + 8x^6) \\ &= 12x^{11} + 60x^9 + 96x^7 + 48x^5. \end{aligned}$$

To compare the two answers, we expand the first one:

$$\begin{aligned} 3(x^4 + 2x^2)^2(4x^3 + 4x) &= 3(x^8 + 4x^6 + 4x^4) \cdot 4(x^3 + x) \\ &= 12(x^{11} + 5x^9 + 8x^7 + 4x^5) \\ &= 12x^{11} + 60x^9 + 96x^7 + 48x^5, \end{aligned}$$

which checks. \blacktriangle

Example 2 Find $\frac{d}{ds}(s^4 + 2s^3 + 3)^8$.

Solution We apply the power of a function rule, with $u = s^4 + 2s^3 + 3$ (and the variable x replaced by s):

$$\begin{aligned} \frac{d}{ds}(s^4 + 2s^3 + 3)^8 &= 8(s^4 + 2s^3 + 3)^7 \frac{d}{ds}(s^4 + 2s^3 + 3) \\ &= 8(s^4 + 2s^3 + 3)^7(4s^3 + 6s^2). \end{aligned}$$

(You could also do this problem by expanding the eighth power and then differentiating; obviously, this practice is not recommended.) \blacktriangle

Example 3 If $y = (x^2 + 1)^{27}(x^4 + 3x + 1)^8$, find the rate of change of y with respect to x .

Solution First of all, by the power of a function rule,

$$\frac{d}{dx}(x^2 + 1)^{27} = 27(x^2 + 1)^{26} \cdot 2x$$

and

$$\frac{d}{dx}(x^4 + 3x + 1)^8 = 8(x^4 + 3x + 1)^7(4x^3 + 3)$$

Thus, by the product rule, the rate of change of y with respect to x is

$$\begin{aligned} \frac{dy}{dx} &= 27(x^2 + 1)^{26} \cdot 2x \cdot (x^4 + 3x + 1)^8 \\ &\quad + (x^2 + 1)^{27} \cdot 8(x^4 + 3x + 1)^7(4x^3 + 3). \end{aligned}$$

To simplify this, we can factor out the highest powers of $x^2 + 1$ and $x^4 + 3x + 1$ to get

$$(x^2 + 1)^{26}(x^4 + 3x + 1)^7[27 \cdot 2x(x^4 + 3x + 1) + (x^2 + 1) \cdot 8(4x^3 + 3)].$$

We can consolidate the expression in square brackets to a single polynomial of

degree 5, getting $2(x^2 + 1)^{26}(x^4 + 3x + 1)^7(43x^5 + 16x^3 + 93x^2 + 27x + 12)$ as our rate of change. [Note: Consult your instructor regarding the amount of simplification required.] ▲

The power of a function rule is a special case of an important differentiation rule called the *chain rule*. To understand this more general rule, we begin by noting that the process of forming the power $[g(x)]^n$ can be broken into two successive operations: first find $u = g(x)$, and then find $f(u)$, where $y = f(u) = u^n$. The chain rule will help us to differentiate any function formed from two functions in this way.

If f and g are functions defined for all real numbers, we define their *composition* to be the function which assigns to x the number $f(g(x))$. The composition is often denoted by $f \circ g$. Thus $(f \circ g)(x) = f(g(x))$. To evaluate $y = (f \circ g)(x)$, we introduce an intermediate variable u and write $u = g(x)$ and $y = f(u)$. To evaluate y , we *substitute* $g(x)$ for u in $f(u)$. (If $f(x)$ and $g(x)$ are not defined for all x , then $(f \circ g)(x)$ is defined only when x is in the domain of g and $g(x)$ is in the domain of f .)

- Example 4** (a) If $f(u) = u^3 + 2$ and $g(x) = (x^2 + 1)^2$, what is $h = f \circ g$?
 (b) Let $f(x) = \sqrt{x}$ and $g(x) = x^3 - 5$. Find $f \circ g$ and $g \circ f$.
 (c) Write $\sqrt{1 + x^2} / [2 + (1 + x^2)^3]$ as a composition of simpler functions.

Solution (a) We calculate $h(x) = f(g(x))$ by writing $u = g(x)$ and substituting in $f(u)$. We get $u = (x^2 + 1)^2$ and so

$$h(x) = f(u) = u^3 + 2 = ((x^2 + 1)^2)^3 + 2 = (x^2 + 1)^6 + 2.$$

$$(b) (f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^3 - 5}.$$

$$(g \circ f)(x) = g(f(x)) = [f(x)]^3 - 5 \\ = (\sqrt{x})^3 - 5 = x^{3/2} - 5.$$

The functions $f \circ g$ and $g \circ f$ are certainly different.

- (c) Let $g(x) = 1 + x^2$ and $f(u) = \sqrt{u} / (2 + u^3)$. Then the given function can be written as $f \circ g$. ▲

Calculator Discussion

On electronic calculators, several functions, such as $1/x$, x^2 , \sqrt{x} , and $\sin x$, are evaluated by the push of a single key. To evaluate the composite function $f \circ g$ on x , you first enter x , then push the key for g to get $g(x)$, then push the key for f to get $f(g(x))$. For instance, let $f(x) = x^2$, $g(x) = \sin x$. To calculate $(f \circ g)(x) = f(g(x)) = (\sin x)^2$ for $x = 32$ (degrees), we enter 32, then press the sin key, then the x^2 key. The result is $32 \rightarrow 0.52991926 \rightarrow 0.28081442$. Notice that $(g \circ f)(x) = \sin(x^2)$ is quite different: entering 32 and pressing the x^2 key followed by the sin key, we get $32 \rightarrow 1024 \rightarrow -0.82903756$. ▲

Do not confuse the composition of functions with the product. We have

$$(fg)(x) = f(x)g(x),$$

while

$$(f \circ g)(x) = f(g(x)).$$

In the case of the product we evaluate $f(x)$ and $g(x)$ separately and then multiply the results; in the case of the composition, we evaluate $g(x)$ first and then apply f to the result. While the order of f and g does not matter for the product, it does for composition.

Composition of Functions

The composition $f \circ g$ is obtained by writing $u = g(x)$ and evaluating $f(u)$. To break up a given function $h(x)$ as a composition, find an intermediate expression $u = g(x)$ such that $h(x)$ can be written in terms of u .

The derivative of a composite function turns out to be the product of the derivatives of the separate functions. The exact statement is given in the following box.

Chain Rule

To differentiate a composition $f(g(x))$, differentiate g at x , differentiate f at $g(x)$, and multiply the results:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

if y is a function of u and u is a function of x .

A complete proof of the chain rule can be given by using the theory of limits (see Review Exercise 99, Chapter 11). The basic argument, however, is simple and goes as follows. If x is changed by a small amount Δx and the corresponding change in $u = g(x)$ is Δu , we know that

$$g'(x) = \frac{du}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

Corresponding to the small change Δu is a change Δy in $y = f(u)$, and

$$f'(u) = \frac{dy}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}.$$

To calculate the rate of change dy/dx , we write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x} = \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) \\ &= \left(\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) = \frac{dy}{du} \frac{du}{dx} = f'(u) \cdot g'(x). \end{aligned}$$

In going to the second line, we replace $\Delta x \rightarrow 0$ by $\Delta u \rightarrow 0$ because the differentiable function g is continuous, i.e., $\Delta x \rightarrow 0$ implies $\Delta u \rightarrow 0$, as we saw in Section 1.3.

There is a flaw in this proof: the Δu determined by Δx could well be zero, and division by zero is not allowed. This difficulty is fortunately not an essential one, and the more technical proof given in Chapter 11 avoids it.

Notice that the chain rule written in Leibniz notation is closely related to

our argument and is easy to remember. Although du does not really have an independent meaning, one may “cancel” it informally from the product $(dy/du) \cdot (du/dx)$ to obtain dy/dx .

A physical model illustrating the chain rule is given at the end of this section.

Example 5 Verify the chain rule for $f(u) = u^2$ and $g(x) = x^3 + 1$.

Solution Let $h(x) = f(g(x)) = [g(x)]^2 = (x^3 + 1)^2 = x^6 + 2x^3 + 1$. Thus $h'(x) = 6x^5 + 6x^2$. On the other hand, since $f'(u) = 2u$ and $g'(x) = 3x^2$,

$$f'(g(x)) \cdot g'(x) = (2 \cdot (x^3 + 1))3x^2 = 6x^5 + 6x^2.$$

Hence the chain rule is verified in this case. \blacktriangle

Let us check that the power of a function rule follows from the chain rule: If $y = [g(x)]^n$, we may write $u = g(x)$, $y = f(u) = u^n$. Since $dy/du = nu^{n-1}$, the chain rule gives

$$\frac{dy}{dx} = f'(g(x))g'(x) = n[g(x)]^{n-1}g'(x).$$

This calculation applies to negative or zero powers as well as positive ones. Thus *the power of a function rule holds for all integer powers.*

Example 6 Let $f(x) = 1/[3x^2 - 2x + 1]^{100}$. Find $f'(x)$.

Solution We write $f(x)$ as $(3x^2 - 2x + 1)^{-100}$. Thus

$$f'(x) = -100(3x^2 - 2x + 1)^{-101}(6x - 2). \blacktriangle$$

The chain rule also solves the problem which began the section.

Example 7 Differentiate $\sqrt{x^3 - 5}$.

Solution In Example 4(b) we saw that $\sqrt{x^3 - 5} = \sqrt{u}$ if $u = x^3 - 5$. Thus, if $y = \sqrt{x^3 - 5} = \sqrt{u}$, then $dy/du = 1/2\sqrt{u}$ (Example 4, Sect. 1.3), and $du/dx = 3x^2$, so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 - 5}}. \blacktriangle$$

Example 8 If $h(x) = f(x^2)$, find a formula for $h'(x)$. Check your formula in the case $f(u) = u^3$.

Solution Let $u = g(x) = x^2$, so $h(x) = f(u)$. Then $h'(x) = f'(u) \cdot g'(x) = f'(x^2) \cdot 2x$. Thus $h'(x) = f'(x^2) \cdot 2x$.

If $f(u) = u^3$, then $f'(u) = 3u^2$ and $f'(x^2) = 3x^4$. Thus, $h'(x) = 3x^4 \cdot 2x = 6x^5$. In fact, $h(x) = (x^2)^3 = x^6$ in this case, so differentiating h directly gives the same result. \blacktriangle

Example 9 Use the chain rule to differentiate $f(x) = ((x^2 + 1)^{20} + 1)^4$. (Do not expand!)

Solution Let $u = (x^2 + 1)^{20} + 1$ and $y = u^4$, so $y = f(x)$. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 4u^3 \frac{du}{dx} = 4((x^2 + 1)^{20} + 1)^3 \frac{du}{dx}.$$

To calculate du/dx we use the chain rule again (or the power of a function rule):

$$\frac{du}{dx} = \frac{d}{dx} [(x^2 + 1)^{20} + 1] = 20(x^2 + 1)^{19} \cdot 2x.$$

Thus

$$\begin{aligned} \frac{dy}{dx} &= 4((x^2 + 1)^{20} + 1)^3 \cdot 20(x^2 + 1)^{19} \cdot 2x \\ &= 160x((x^2 + 1)^{20} + 1)^3 (x^2 + 1)^{19}. \blacktriangle \end{aligned}$$

The following examples require us to translate words into equations before using the chain rule.

Example 10 The population of Thin City is increasing at the rate of 10,000 people per day on March 30, 1984. The area of the city grows to keep the ratio of 1 square mile per 1000 people. How fast is the area increasing per day on this date?

Solution Let A = area, p = population, t = time (days). The rate of increase of area with respect to time is $dA/dt = (dA/dp)(dp/dt) = \frac{1}{1000} \cdot 10000 = 10$ square miles per day. \blacktriangle

Example 11 A dog 2 feet high trots proudly away from a 10-foot-high light post. When he is 8 feet from the post's base, he is moving at 3 feet per second. How fast is the tip of his shadow moving?

Solution Refer to Fig. 2.2.1. By similar triangles, $y/(y - x) = 10/2$; solving, $y = 5x/4$. Then $dy/dt = (dy/dx)(dx/dt) = (\frac{5}{4})3 = 3\frac{3}{4}$ feet per second. \blacktriangle

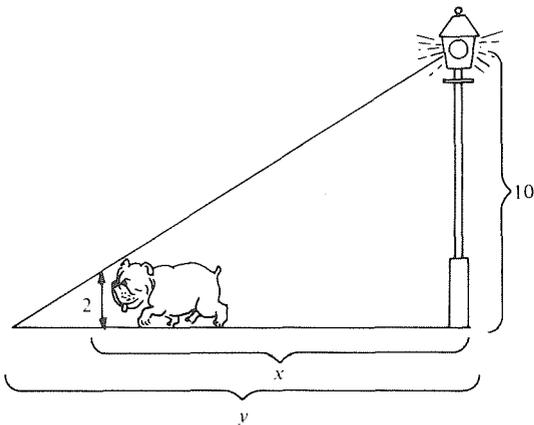


Figure 2.2.1. Dog trotting proudly away from lamp post.

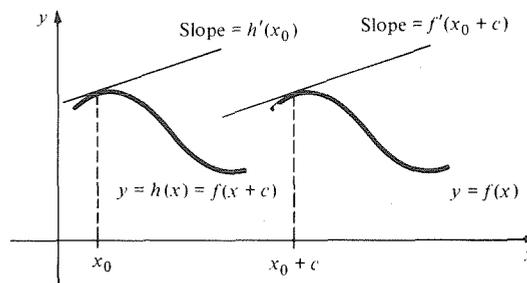


Figure 2.2.2. The geometric interpretation of the shifting rule.

Another special case of the chain rule may help you to understand it. Consider $h(x) = f(x + c)$, c a constant. If we let $u = g(x) = x + c$, we get $g'(x) = 1$, so

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(x + c) \cdot 1 = f'(x + c).$$

Note that the graph of h is the same as that of f except that it is shifted c units to the left (see Fig. 2.2.2). It is reasonable, then, that the tangent line to the graph of h is obtained by shifting the tangent line to the graph of f . Thus, in this case, the chain rule is telling us something geometrically obvious. One might call this formula the *shifting rule*. In Leibniz notation it reads

$$\left. \frac{d}{dx} f(x + c) \right|_{x_0} = \left. \frac{d}{dx} f(x) \right|_{x_0 + c}.$$

Supplement to Section 2.2 A Physical Model for the Chain Rule

When you change altitude rapidly, as in a moving car or plane, a pressure difference develops between the inside and outside of your eardrums, and your ears “pop.” Three variables relevant to this phenomenon are the time t , the altitude u , and the air pressure p . Ear popping occurs when the rate of change dp/dt is too large.

The rate dp/dt is hard to measure directly. On the other hand, du/dt can be determined if we know the altitude as a function of t . For instance, if we are rolling down a hill at 100 kilometers per hour, we could have $du/dt = -3$ meters per second. The rate of change dp/du is known to meteorologists; near sea level, it is about -0.12 gsc per meter. (The unit “gsc” of pressure is “grams per square centimeter”; the rate of change is negative because pressure decreases as altitude increases.)

Now the chain rule enables us to calculate how fast the pressure is changing with time:

$$\frac{dp}{dt} = \frac{dp}{du} \cdot \frac{du}{dt} = (-0.12)(-3) \text{ gsc per second} = 0.36 \text{ gsc per second.}$$

This rate of pressure increase is fast enough so that the ears’ internal pressure control system cannot keep up with it, and they “pop.”

Exercises for Section 2.2

Find the derivatives of the functions in Exercises 1–10.

- $(x + 3)^4$
- $(x^2 + 3x + 1)^5$
- $(x^3 + 10x)^{100}$
- $(s^4 + 4s^3 + 3s^2 + 2s + 1)^8$
- $(x^2 + 8x)^3 \cdot x$
- $(x^2 + 2)^3(x^9 + 8)$
- $(x^2 + 2)(x^9 + 8)^3$
- $(x^3 + 2)^9(x^8 + 2x + 1)^{10}$
- $(y + 1)^3(y + 2)^2(y + 3)$
- $((x^2 - 1)^2 + 3)^{10}$

11. Let $g(x) = x + 1$ and $f(u) = u^2$. Find $f \circ g$ and $g \circ f$.

12. Let $h(x) = x^{24} + 3x^{12} + 1$. Write $h(x)$ as a composite function $f(g(x))$ with $g(x) = x^{12}$.

Find $f \circ g$ and $g \circ f$ in each of Exercises 13–16.

13. $g(x) = x^3$; $f(x) = (x - 2)^3$.

14. $g(x) = x^n$; $f(x) = x^m$.

15. $g(x) = \frac{1}{1-x}$; $f(x) = \frac{1}{2} - \sqrt{3}x$.

16. $g(x) = \frac{3x-2}{4x+1}$; $f(x) = \frac{2x-7}{9x+3}$.

Write the functions in Exercises 17–20 as compositions of simpler functions.

17. $h(x) = \sqrt{4x^3 + 5x + 3}$.

18. $h(r) = \sqrt{1 + \sqrt{r}}$.

19. $h(u) = \left(\frac{1-u}{1+u}\right)^3$.

20. $h(x) = ((x^2 + 1) + (x^2 + 1)^2 + 1)^2$.

Verify the chain rule for $f(u)$ and $g(x)$ given in Exercises 21–24.

21. $f(u) = u^2$, $g(x) = x^2 - 1$

22. $f(u) = u^3$, $g(x) = x + 1$

23. $f(u) = u^2$, $g(x) = \sqrt{x}$

24. $f(u) = \sqrt{u}$, $g(x) = x^2$

Use the chain rule to differentiate the functions in Exercises 25–34.

25. $(x^2 - 6x + 1)^3$

26. $(x - 2x^2)^3$

27. $\frac{9 + 2x^5}{3 + 5x^5}$

28. $\frac{1}{(x^3 + 5x)^4}$

29. $((x^2 + 2)^2 + 1)^2$

30. $\frac{3}{[(x + 2)^2 + 4]^4}$

31. $\frac{(x^2 + 3)^5}{[1 + (x^2 + 3)]^8}$

32. $\frac{1}{2x + 1} [(2x + 1)^3 + 5]$

33. $\sqrt{4x^5 + 5x^2}$

34. $\sqrt{1 + \sqrt{x}}$

35. If $h(x) = x^3 f(2x^2)$, find a formula for $h'(x)$.

36. If $h(x) = f(g(x^2))$, find a formula for $h'(x)$.

37. Given three functions, f , g , and h :

(a) How would you define the composition $f \circ g \circ h$?

(b) Use the chain rule twice to obtain a formula for the derivative of $f \circ g \circ h$.

38. If $h(x) = f(g(x^3 + 2)) + g(f(x^2))$, find a formula for $h'(x)$.

39. Fat City occupies a circular area 10 miles in diameter and contains 500,000 inhabitants. If the population is growing now at the rate of 20,000 inhabitants per year, how fast should the diame-

ter be increasing now in order to maintain the circular shape and the same population density (= number of people per square mile). If the population continues to grow at the rate of 20,000 per year, how fast should the diameter be increasing in 5 years? Give an intuitive explanation of the relation between the two answers.

40. The radius at time t of a sphere S is given by $r = t^2 - 2t + 1$. How fast is the volume V of S changing at time $t = \frac{1}{2}, 1, 2$?
41. The kinetic energy K of a particle of mass m moving with speed v is $K = \frac{1}{2}mv^2$. A particle with mass 10 grams has, at a certain moment, velocity 30 centimeters per second and acceleration 5 centimeters per second per second. At what rate is the kinetic energy changing?
42. (a) At a certain moment, an airplane is at an altitude of 1500 meters and is climbing at the rate of 5 meters per second. At this altitude, pressure decreases with altitude at the rate of 0.095 gsc per meter. What is the rate of change of pressure with respect to time?
- (b) Suppose that the airplane in (a) is descending rather than climbing at the rate of 5 meters per second. What is the rate of change of pressure with respect to time?
43. At a certain moment, your car is consuming gasoline at the rate of 15 miles per gallon. If gasoline costs 75 cents per gallon, what is the cost per mile? Set the problem up in terms of functions and apply the chain rule.
44. The price of eggs, in cents per dozen, is given by the formula $p = 55/(s - 1)^2$, where s is the supply of eggs, in units of 10,000 dozen, available to the wholesaler. Suppose that the supply on July 1, 1986 is $s = 2.1$ and is falling at a rate of 0.03 per month. How fast is the price rising?
45. If an object has position $(t^2 + 4)^5$ at time t , what is its velocity when $t = -1$?
46. If an object has position $(t^2 + 1)/(t^2 - 1)$ at time t , what is its velocity when $t = 2$?

Find the second derivatives of the functions in Exercises 47–50.

47. $(x + 1)^{13}$
 48. $(x^3 - 1)^8$
 49. $(x^4 + 10x^2 + 1)^{98}$
 50. $(x^2 + 1)^3(x^3 + 1)^2$

- ★51. (a) Find a “stretching rule” for the derivative of $f(cx)$, c a constant.
 (b) Draw the graphs of $y = 1 + x^2$ and of $y = 1 + (4x)^2$ and interpret the stretching rule geometrically.

★52. Prove that $(d/dx)(u^n) = nu^{n-1}du/dx$ for all natural numbers n as follows:

- (a) Note that this is established for $n = 1, 2, 3$ at the beginning of this section.
 (b) Assume that the result is true for $n - 1$, and write $u^n = u(u^{n-1})$. Now differentiate using the product rule to establish the result for n .
 (c) Use induction to conclude the result for all n . (See Exercise 65, p. 69.)

★53. Find a general formula for $(d^2/dx^2)(u^n)$, where $u = f(x)$ is any function of x .

★54. (a) Let i be the “identity function” $i(x) = x$. Show that $i \circ f = f$ and $f \circ i = f$ for any function f . (b) Verify the chain rule for $f \circ i$.

★55. Let f and g be functions such that $f \circ g = i$, where i is the function in Problem 54. Find a formula for $f'(x)$ in terms of the derivative of g .

★56. Use the result of Exercise 55 to find the derivative of $f(x) = \sqrt[3]{x}$ by letting $g(x) = x^3$.

★57. Find a formula for the second derivative of $f \circ g$ in terms of the first and second derivatives of f and g .

★58. Show that the power of a function rule for negative powers follows from that rule for positive powers and the reciprocal rule.

★59. For reasons which will become clear in Chapter 6, the quotient $f'(x)/f(x)$ is called the *logarithmic derivative* of $f(x)$.

(a) Show that the logarithmic derivative of the product of two functions is the sum of the logarithmic derivatives of the functions.

(b) Show that the logarithmic derivative of the quotient of two functions is the difference of their logarithmic derivatives.

(c) Show that the logarithmic derivative of the n th power of a function is n times the logarithmic derivative of the function.

(d) Develop a formula for the logarithmic derivative of

$$[f_1(x)]^{n_1}[f_2(x)]^{n_2} \cdots [f_k(x)]^{n_k}$$

in terms of the logarithmic derivatives of f_1 through f_k .

(e) Using your formula in part (d), find the ordinary (not logarithmic) derivative of

$$f(x) = \frac{(x^2 + 3)(x + 4)^8(x + 7)^9}{(x^4 + 3)^{17}(x^4 + 2x + 1)^5}$$

If you have enough stamina, compute $f'(x)$ without using the formula in part (d).

★60. Differentiate $(1 + (1 + (1 + x^2)^8)^8)^8$.

2.3 Fractional Powers and Implicit Differentiation

The power rule still holds when the exponent is a fraction.

In this section we extend the power rule to include fractional exponents by using a method called implicit differentiation, which can be applied to many other problems as well.

Let us begin by trying to find dy/dx when $y = x^{1/n} = \sqrt[n]{x}$, where n is a positive integer.¹ At the moment, we shall simply assume that this derivative exists and try to calculate its value. This assumption will be justified in Section 5.3, in connection with inverse functions.

We may rewrite the relation $y = x^{1/n}$ as $y^n = x$, so we must have

$$\frac{d}{dx}(y^n) = \frac{d}{dx}(x). \quad (1)$$

Recalling that y is a function of x , we may evaluate the left-hand side of (1) by the chain rule (or the power of a function rule) to get

$$\frac{d}{dx}(y^n) = ny^{n-1} \frac{dy}{dx}. \quad (2)$$

The right-hand side of (1) is simply

$$\frac{dx}{dx} = 1. \quad (3)$$

Substituting (2) and (3) into (1) gives

$$ny^{n-1} \frac{dy}{dx} = 1$$

which we may solve for dy/dx to obtain

$$\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{n} y^{1-n} = \frac{1}{n} (x^{1/n})^{1-n} = \frac{1}{n} x^{(1-n)/n} = \frac{1}{n} x^{(1/n)-1}.$$

Thus

$$\frac{d}{dx}(x^{1/n}) = \frac{1}{n} x^{(1/n)-1}. \quad (4)$$

Note that this rule reads the same as the ordinary power rule: "Bring down the exponent as a multiplier and then decrease the exponent by one." The special case $(d/dx)(x^{1/2}) = \frac{1}{2}x^{-1/2}$ has already been considered in Example 4, Section 1.3.

Example 1 Differentiate $f(x) = 3\sqrt[5]{x}$.

Solution $\frac{d}{dx} 3\sqrt[5]{x} = 3 \frac{d}{dx} x^{1/5} = \frac{3}{5} x^{(1/5)-1} = \frac{3}{5} x^{-4/5} = \frac{3}{5x^{4/5}}. \blacktriangle$

Next, we consider a general rational power $f(x) = x^r$, where $r = p/q$ is a ratio of integers. Thinking of $x^{p/q}$ as $(x^{1/q})^p$, we set $g(x) = x^{1/q}$, so that $f(x) = [g(x)]^p$. Then, by the (integer) power of a function rule,

$$\frac{d}{dx} [g(x)]^p = p[g(x)]^{p-1} g'(x),$$

¹Note that $x^{1/n}$ is defined for all x if n is odd but only for nonnegative x if n is even. A brief review of fractional exponents may be found in Section R.3.

so by formula (4) with $1/n$ replaced by $1/q$, we have

$$\begin{aligned}\frac{d}{dx}(x^{p/q}) &= \frac{d}{dx}(x^{1/q})^p = p(x^{1/q})^{p-1} \cdot \frac{1}{q} x^{(1/q)-1} \\ &= \frac{p}{q} x^{(p-1)/q} x^{(1-q)/q} = \frac{p}{q} x^{(p-q)/q} = \frac{p}{q} x^{(p/q)-1}.\end{aligned}$$

We conclude that differentiation of rational powers follows the same rule as integer powers.

Rational Power Rule

To differentiate a power x^r (r a rational number), take out the exponent as a factor and then reduce the exponent by 1:

$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

(The formula is valid for all x for which the right-hand side makes sense.)

Example 2 Differentiate $f(x) = 3x^2 + (x^2 + x^{1/3})/\sqrt{x}$.

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}(3x^2 + x^{3/2} + x^{-1/6}) \\ &= 6x + \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-7/6} \\ &= 6x + \frac{3}{2}\sqrt{x} - \frac{1}{6x^{7/6}} \blacktriangle\end{aligned}$$

We can combine the rational power rule with the chain rule to prove a rational power of a function rule. Let $y = [f(x)]^r$ and let $u = f(x)$ so that $y = u^r$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = ru^{r-1} \frac{du}{dx} = r[f(x)]^{r-1} f'(x).$$

Rational Power of a Function Rule

To differentiate a power $[f(x)]^r$ (r a rational number), take out the exponent as a factor, reduce the exponent by 1, and multiply by $f'(x)$:

$$\frac{d}{dx}[f(x)]^r = r[f(x)]^{r-1} f'(x).$$

Example 3 Differentiate $g(x) = (9x^3 + 10)^{5/3}$.

Solution Here $f(x) = 9x^3 + 10$, $r = \frac{5}{3}$, and $f'(x) = 27x^2$. Thus

$$g'(x) = \frac{5}{3}(9x^3 + 10)^{2/3} \cdot 27x^2 = 45x^2(9x^3 + 10)^{2/3}. \blacktriangle$$

The rules for rational powers can be combined with the quotient rule of differentiation, as in the next example.

Example 4 Differentiate $\frac{x^{1/2} + x^{3/2}}{x^{3/2} + 1}$.

Solution We use the quotient and rational power rules:

$$\begin{aligned} \frac{d}{dx} \left[\frac{x^{1/2} + x^{3/2}}{x^{3/2} + 1} \right] &= \frac{[x^{3/2} + 1] (d/dx) [x^{1/2} + x^{3/2}] - [x^{1/2} + x^{3/2}] (d/dx) [x^{3/2} + 1]}{[x^{3/2} + 1]^2} \\ &= \frac{[x^{3/2} + 1] \left[\frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2} \right] - [x^{1/2} + x^{3/2}] \left[\frac{3}{2}x^{1/2} \right]}{x^3 + 2x^{3/2} + 1} \\ &= \frac{\left(\frac{1}{2}x + \frac{3}{2}x^2 + \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2} \right) - \left(\frac{3}{2}x + \frac{3}{2}x^2 \right)}{x^3 + 2x^{3/2} + 1} \\ &= \frac{3x^{1/2} + x^{-1/2} - 2x}{2(x^3 + 2x^{3/2} + 1)} \cdot \blacktriangle \end{aligned}$$

The method which we used to differentiate $y = x^{1/n}$, namely differentiating the relation $y^n = x$ and then solving for dy/dx , is called *implicit differentiation*. This method can be applied to more complicated relationships such as $x^2 + y^2 = 1$ or $x^4 + xy + y^5 = 2$ which define y as a function of x *implicitly* rather than *explicitly*. In general, such a relationship will not define y uniquely as a function of x ; it may define two or more functions. For example, the circle $x^2 + y^2 = 1$ is not the graph of a function, but the upper and lower semicircles are graphs of functions (see Fig. 2.3.1).

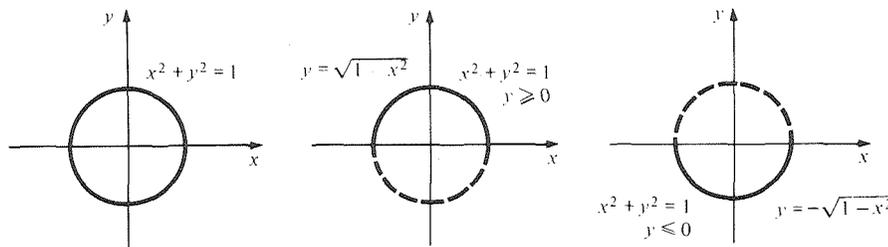


Figure 2.3.1. Parts of the circle $x^2 + y^2 = 1$ are the graphs of functions.

Example 5 If $y = f(x)$ and $x^2 + y^2 = 1$, express dy/dx in terms of x and y .

Solution Thinking of y as a function of x , we differentiate both sides of the relation $x^2 + y^2 = 1$ with respect to x . The derivative of the left-hand side is

$$\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} x^2 + \frac{d}{dx} y^2 = 2x + 2y \frac{dy}{dx},$$

while the right-hand side has derivative zero. Thus

$$2x + 2y \frac{dy}{dx} = 0 \quad \text{and so} \quad \frac{dy}{dx} = -\frac{x}{y} \cdot \blacktriangle$$

The result of Example 5 can be checked, since in this case we can solve for y directly:

$$y = \pm \sqrt{1 - x^2}.$$

Notice that the given relation then defines *two* functions: $f_1(x) = \sqrt{1 - x^2}$ and

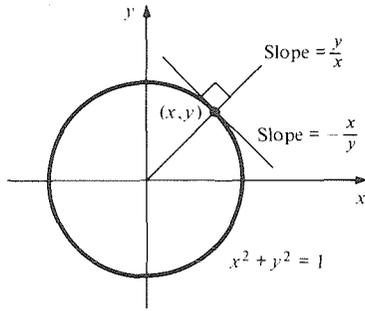
$f_2(x) = -\sqrt{1-x^2}$. Taking the plus case, with $u = 1-x^2$ and $y = \sqrt{u}$, the chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}} (-2x) \\ &= \frac{1}{2\sqrt{1-x^2}} (-2x) = \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}, \end{aligned}$$

so it checks. The minus case gives the same answer.

From the form of the derivative given by implicit differentiation, $dy/dx = -x/y$, we see that the tangent line to a circle at (x, y) is perpendicular to the line through (x, y) and the origin, since their slopes are negative reciprocals of one another. (See Fig. 2.3.2.) Implicit differentiation often leads directly to such striking results, and for this reason it is sometimes preferable to use this method even when y could be expressed in terms of x .

Figure 2.3.2. If $x^2 + y^2 = 1$, the formula $dy/dx = -x/y$ means that the tangent line to a circle at a point on the circle is perpendicular to the line from that point to the center of the circle.



There is a device which may help you to remember that the chain rule must be used. In Example 5, if we keep the notation $f(x)$ for y , then the relation $x^2 + y^2 = 1$ becomes $x^2 + [f(x)]^2 = 1$, and differentiating with respect to x gives $2x + 2f(x)f'(x) = 0$. Now we solve for $f'(x)$ to get $f'(x) = -x/f(x)$ or, in Leibniz notation, $dy/dx = -x/y$, just as before. Once you have done a few examples in this long-winded way, you should be able to go back to y and dy/dx without the f .

The following is an example in which we cannot solve for y in terms of x .

Example 6 Find the equation of the tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$.

Solution We note first that $(1, 2)$ lies on the curve, since $2(1)^6 + 2^4 = 9(1)(2)$. Now suppose that $y = f(x)$ and differentiate both sides of the defining relation. The left-hand side gives

$$\frac{d}{dx} (2x^6 + y^4) = 12x^5 + 4y^3 \frac{dy}{dx},$$

while the right-hand side gives

$$\frac{d}{dx} (9xy) = 9y + 9x \frac{dy}{dx}.$$

Equating both sides and solving for dy/dx , we have

$$12x^5 + 4y^3 \frac{dy}{dx} = 9y + 9x \frac{dy}{dx},$$

$$(4y^3 - 9x) \frac{dy}{dx} = 9y - 12x^5,$$

$$\frac{dy}{dx} = \frac{9y - 12x^5}{4y^3 - 9x}.$$

When $x = 1$ and $y = 2$,

$$\frac{dy}{dx} = \frac{9(2) - 12(1)^3}{4(2)^3 - 9(1)} = \frac{18 - 12}{32 - 9} = \frac{6}{23}.$$

Thus, the slope of the tangent line is $\frac{6}{23}$; by the point-slope formula, the equation of the tangent line is $y - 2 = \frac{6}{23}(x - 1)$, or $y = \frac{6}{23}x + \frac{40}{23}$. \blacktriangle

Implicit Differentiation

To calculate dy/dx if x and y are related by an equation:

1. Differentiate both sides of the equation with respect to x , thinking of y as a function of x and using the chain rule.
2. Solve the resulting equation for dy/dx .

Exercises for Section 2.3

Differentiate the functions in Exercises 1–24.

- | | |
|--|--|
| 1. $10x^{1/8}$ | 2. $x^{3/5}$ |
| 3. $8x^{1/4} - x^{-2/3}$ | 4. $8x^4 - 3x^{-5/4}$ |
| 5. $3x^{2/3} - (5x)^{1/2}$ | 6. $x^2 - 3x^{1/2}$ |
| 7. $x^2(x^{1/3} + x^{4/3})$ | 8. $(x + 2)^{3/2}\sqrt{x + 2}$ |
| 9. $(x^5 + 1)^{7/9}$ | 10. $(x^{1/3} + x^{2/3})^{1/3}$ |
| 11. $\frac{1}{\sqrt{x}}$ | 12. $\frac{1}{\sqrt[4]{x^3 + 5x + 2}}$ |
| 13. $\frac{\sqrt{x^2 + 1}}{\sqrt{x^2 - 1}}$ | 14. $\frac{x^{1/7}}{(3x^{5/3} + x)}$ |
| 15. $\sqrt{(x + 3)^2 - 4}$ | 16. $3\sqrt{x} - \left(\frac{1}{x}\right) + 5$ |
| 17. $\frac{\sqrt{x}}{3 + x + x^3}$ | 18. $\frac{x}{\sqrt{x^2 + 2}}$ |
| 19. $\frac{\sqrt{x}}{1 + \sqrt{x}}$ | 20. $\frac{\sqrt{x}}{\sqrt[3]{x^4 + 2}}$ |
| 21. $\frac{\sqrt[3]{x}}{x^2 + 2}$ | 22. $\frac{(3x^2 - 6x)}{3\sqrt[4]{x}}$ |
| 23. $\sqrt{\frac{6x^2 + 2x + 1}{\sqrt{x} + 2x^3}}$ | 24. $\sqrt{\frac{\sqrt[3]{x}}{\sqrt{3x^2 + 1 + x}}}$ |

Find the indicated derivatives in Exercises 25–28.

25. $\left(\frac{d}{dx}\right)(x^2 + 5)^{7/8}$
26. $\left(\frac{d^2}{dx^2}\right)(x^r)$, where r is rational
27. $f'(7)$, where $f(x) = 7\sqrt[3]{x}$
28. $\left(\frac{d}{dx}\right)\left(x^{1/4} + \frac{4}{\sqrt{x}}\right)\Big|_{x=81}$

Find the derivatives of each of the functions in Exercises 29–34.

29. $f(x) = x^{3/11} - x^{1/5}$
30. $k(s) = \frac{1}{s^{3/5} - s}$
31. $h(y) = \frac{y^{1/8}}{y - 2}$
32. $g(t) = t(t^{2/3} + t^7)$

$$33. l(x) = \left[\frac{x^{1/2} + 1}{x^{1/2} - 1}\right]^{1/2}$$

$$34. m(u) = (u^9 - 1)^{-6/7}$$

35. If $x^2 + y^2 = 3$, compute dy/dx when $x = 0$ and $y = \sqrt{3}$.

36. If $x^3 + y^3 = xy$, compute dx/dy in terms of x and y .

37. Suppose that $x^4 + y^2 + y - 3 = 0$.

(a) Compute dy/dx by implicit differentiation.

(b) What is dy/dx when $x = 1$, $y = 1$?

(c) Solve for y in terms of x (by the quadratic formula) and compute dy/dx directly. Compare with your answer in part (a).

38. Suppose that $xy + \sqrt{x^2 - y} = 7$.

(a) Find dy/dx .

(b) Find dx/dy .

(c) What is the relation between dy/dx and dx/dy ?

39. Suppose that $x^2/(x + y^2) = y^2/2$.

(a) Find dy/dx when $x = 2$, $y = \sqrt{2}$.

(b) Find dy/dx when $x = 2$, $y = -\sqrt{2}$.

40. Let $(u^2 + 6)(v^2 + 1) = 10uv$. Find du/dv and dv/du when $u = 2$ and $v = 1$.

41. Find the equation of the tangent line to the curve $x^4 + y^4 = 2$ when $x = y = 1$.

42. Find the equation of the tangent line to the curve $2x^2 + 2xy + y^2 = 8$ when $x = 2$ and $y = 0$.

43. Find $(d^2/dx^2)(x^{1/2} - x^{2/3})$.

44. Find $(d^2/dx^2)(x/\sqrt{1 + x^2})$.

45. Find the equation of the tangent line to the graph of $y = \sqrt{1 - x^2}$ at the point $(\sqrt{3}/2, 1/2)$.

46. Find the equation of the line tangent to $y = (x^{1/2} + x^{1/3})^{1/3}$ at $x = 1$.

47. Let $x^4 + y^4 = 1$. Find dy/dx as a function of x in two ways: by implicit differentiation and by solving for y in terms of x .

48. Differentiate the function $(x^3 + 2)/\sqrt{x^3 + 1}$ in at least two different ways. Be sure that the answers you get are equivalent. (Doing the same problem in several ways is a good method for checking your calculations, useful on examinations as well as in scientific work.)

Find linear approximations for the expressions in Exercises 49–52.

49. $\sqrt[4]{15.97}$ 50. $(4.02)^{3/2}$
 51. $(-26.98)^{2/3}$ 52. $\sqrt[3]{122}$
53. The mass M of the first x meters of a concrete beam is $M = 24(2 + x^{1/3})^4$ kilograms. Find the density dM/dx .
54. The average pulse rate for persons 30 inches to 74 inches tall can be approximated by $y = 589/\sqrt{x}$ beats per minute for a person x inches tall.
 (a) Find dy/dx , and show that it is always negative.
 (b) The value of $|dy/dx|$ at $x = 65$ is the decrease in beats per minute expected for a 1-inch increase in height. Explain.
 (c) Do children have higher pulse rates than adults according to this model?
55. Let $y = 24\sqrt{x}$ be the learning curve for learning y items in x hours, $0 \leq x \leq 5$. Apply the linear approximation to estimate the number of new items learned in the 12-minute period given by $1 < x < 1.2$.

56. The daily demand function for a certain manufactured good is $p(x) = 75 + 4\sqrt{2x} - (x^2/2)$, where x is the production level.
 (a) Find dp/dx . Interpret.
 (b) Find the production level x for which $dp/dx = 0$. Interpret.
57. The fundamental period for vibration P and the tension T in a certain string are related by $P = \sqrt{32/T}$ seconds. Find the rate of change of period with respect to tension when $T = 9$ lbs.
58. Lions in a small district in an African game preserve defend an exclusive region of area A which depends on their body weight W by the formula $A = W^{1.31}$.
 (a) Find dA/dW .
 (b) By what percentage should the defended area increase after a 200-lb lion undergoes a 20-lb weight gain? (Use the linear approximation.)
59. The object distance x and image distance y satisfy the thin lens equation $1/x + 1/y = 1/f$, where f is the focal length.
 (a) Solve for y as a function of x when $f = 50$ millimeters.
 (b) Find dy/dx .
 (c) Find all (x, y) such that $(d/dx)(x + y) = 0$.
- *60. Using implicit differentiation, find the equation of the tangent line at the point (x_0, y_0) on the circle $(x - a)^2 + (y - b)^2 = r^2$. Interpret your result geometrically. (a , b , and r are constants.)

2.4 Related Rates and Parametric Curves

If two quantities satisfy an equation, their rates of change can be related by implicit differentiation.

Suppose that we have two quantities, x and y , each of which is a function of time t . We know that the rates of change of x and y are given by dx/dt and dy/dt . If x and y satisfy an equation, such as $x^2 + y^2 = 1$ or $x^2 + y^6 + 2y = 5$, then the rates dx/dt and dy/dt can be related by differentiating the equation with respect to t and using the chain rule.

Example 1 Suppose that x and y are functions of t and that $x^4 + xy + y^4 = 1$. Relate dx/dt and dy/dt .

Solution Differentiate the relation between x and y with respect to t , thinking of x and y as functions of t :

$$\begin{aligned} \frac{d}{dt}(x^4 + xy + y^4) &= 0, \\ 4x^3 \frac{dx}{dt} + \frac{dx}{dt}y + x \frac{dy}{dt} + 4y^3 \frac{dy}{dt} &= 0. \end{aligned}$$

We can simplify this to

$$\frac{dy}{dt} = -\left(\frac{4x^3 + y}{x + 4y^3}\right) \frac{dx}{dt}$$

which is the desired relation. ▲

If you have trouble remembering to use the chain rule, you can use a device like that following Example 5 in Section 2.3. Namely, write $f(t)$ for x and $g(t)$ for y , then differentiate the relation (such as $[f(t)]^4 + f(t)g(t) + [g(t)]^4 = 1$) with respect to t . This will give a relation between $f'(t)$, $g'(t)$, $f(t)$, and $g(t)$. Once you have done a few examples in this long-winded way, you should be ready to go back to the d/dt 's.

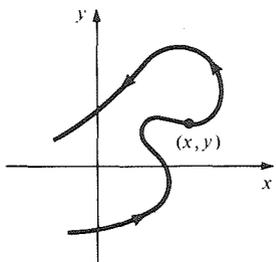


Figure 2.4.1. If x and y are functions of t , the point (x, y) follows a curve as t varies.

Related Rates

To relate the rates dx/dt and dy/dt if x and y satisfy a given equation:

1. Differentiate both sides of the equation with respect to t , thinking of x and y as functions of t .
2. Solve the resulting equation for dy/dt in terms of dx/dt (or vice versa if called for).

There is a useful geometric interpretation of related rates. (This topic is treated in more detail in Section 10.4.) If x and y are each functions of t , say $x = f(t)$ and $y = g(t)$, we can plot the points (x, y) for various values of t . As t varies, the point (x, y) will move along a curve. When a curve is described this way, it is called a *parametric curve* (see Fig. 2.4.1).

Example 2 If $x = t^4$ and $y = t^2$, what curve does (x, y) follow for $-\infty < t < \infty$?

Solution We notice that $y^2 = x$, so the point (x, y) lies on a parabola. As t ranges from $-\infty$ to ∞ , y goes from $+\infty$ to zero and back to $+\infty$, so (x, y) stays on the half of the parabola with $y \geq 0$ and traverses it twice (see Fig. 2.4.2). ▲

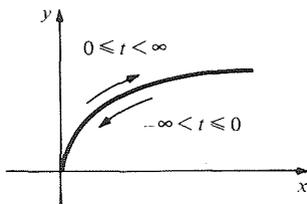


Figure 2.4.2. As t ranges from $-\infty$ to $+\infty$, the point (t^4, t^2) traverses the parabola twice in the directions shown.

It may be possible to describe a parametric curve in other ways. For instance, it may be described by a relation between x and y . Specifically, suppose that the parametric curve $x = f(t)$, $y = g(t)$ can be described by an equation $y = h(x)$ (the case $x = k(y)$ will be similar). Then we can differentiate by the chain rule. Using Leibniz notation:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \quad \text{so} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

This shows that the slope of the tangent line to a parametric curve is given by $(dy/dt)/(dx/dt)$.

Parametric Curves

As t varies, two equations $x = f(t)$ and $y = g(t)$ describe a curve in the plane called a parametric curve. The slope of its tangent line is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

Example 3 Find the equation of the line tangent to the parametric curve given by the equations $x = (1 + t^3)^4 + t^2$, $y = t^5 + t^2 + 2$ at $t = 1$.

Solution Here the relation between x and y is not clear, but we do not need to know it. (We tacitly assume that the path followed by (x, y) can be described by a function $y = h(x)$.) We have

$$\frac{dx}{dt} = 4(1 + t^3)^3 \cdot 3t^2 + 2t = 12(1 + t^3)^3 t^2 + 2t \quad \text{and} \quad \frac{dy}{dt} = 5t^4 + 2t,$$

so the slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5t^4 + 2t}{12t^2(1 + t^3)^3 + 2t} = \frac{5t^3 + 2}{12t(1 + t^3)^3 + 2}.$$

At $t = 1$, we get

$$\frac{dy}{dx} = \frac{7}{12 \cdot 2^3 + 2} = \frac{7}{98} = \frac{1}{14}.$$

Since $x = 17$ and $y = 4$ at $t = 1$, the equation of the tangent line is given by the point-slope formula:

$$y - 4 = \frac{1}{14}(x - 17),$$

$$y = \frac{x}{14} + \frac{39}{14}. \quad \blacktriangle$$

Example 4 Show that the parametric equations $x = at + b$ and $y = ct + d$ describe a straight line if a and c are not both zero. What is its slope?

Solution Multiplying $x = at + b$ by c , multiplying $y = ct + d$ by a , and subtracting, we get

$$cx - ay = bc - ad,$$

so $y = (c/a)x + (1/a)(ad - bc)$, which is the equation of a line with slope c/a . (If $a = 0$, x is constant and the line is vertical; if c were also zero the line would degenerate to a point.) Note that the slope can also be obtained as $(dy/dt)/(dx/dt)$, since $dy/dt = c$ and $dx/dt = a$. \blacktriangle

Example 5 Suppose that x and y are functions of time and that (x, y) moves on the circle $x^2 + y^2 = 1$. If x is increasing at 1 centimeter per second, what is the rate of change of y when $x = 1/\sqrt{2}$ and $y = 1/\sqrt{2}$?

Solution Differentiating $x^2 + y^2 = 1$ gives $2x(dx/dt) + 2y(dy/dt) = 0$; so $dy/dt = (-x/y)(dx/dt)$. If $x = y = 1/\sqrt{2}$, $dy/dt = -dx/dt = -1$ centimeter per second. \blacktriangle

In word problems involving related rates, the hardest job may be to translate the verbal problem into mathematical terms. You need to identify the variables which are changing with time and to find relations between them. If some geometry is involved, drawing a figure is essential and will often help you to spot the important relations. Similar triangles and Pythagoras' theorem are frequently useful in these problems.

Example 6 A light L is being raised up a pole (see Fig. 2.4.3). The light shines on the object Q , casting a shadow on the ground. At a certain moment the light is 40 meters off the ground, rising at 5 meters per minute. How fast is the shadow shrinking at that instant?

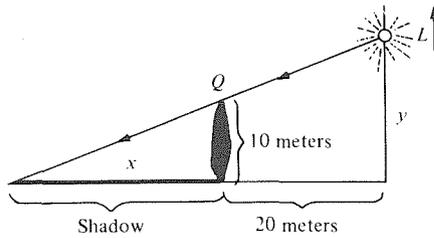


Figure 2.4.3. At what rate is the shadow shrinking?

Solution Let the height of the light be y at time t and the length of the shadow be x . By similar triangles, $x/10 = (x + 20)/y$; i.e., $xy = 10(x + 20)$. Differentiating with respect to t , $x(dy/dt) + (dx/dt)y = 10(dx/dt)$. At the moment in question $y = 40$, and so $x \cdot 40 = 10(x + 20)$ or $x = \frac{20}{3}$. Also, $dy/dt = 5$ and so $\frac{20}{3} \cdot 5 + 40(dx/dt) = 10(dx/dt)$. Solving for (dx/dt) , we get $dx/dt = -\frac{10}{9}$. Thus the shadow is shrinking at $\frac{10}{9}$ meters per minute. ▲

Example 7 A spherical balloon is being blown up by a child. At a certain instant during inflation, air enters the balloon to make the volume increase at a rate of 50 cubic centimeters a second. At the same instant the balloon has a radius of 10 centimeters. How fast is the radius changing with time?

Solution Let the radius of the balloon be denoted by r and the volume by V . Thus $V = \frac{4}{3}\pi r^3$ and so

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

At the instant in question, $dV/dt = 50$ and $r = 10$. Thus

$$50 = 4\pi(10)^2 \frac{dr}{dt},$$

and so

$$\frac{dr}{dt} = \frac{50}{400\pi} = \frac{1}{8\pi} \approx 0.04 \text{ centimeters per second. } \blacktriangle$$

Example 8 A thunderstorm is dropping rain at the rate of 2 inches per hour into a conical tank of diameter 15 feet and height 30 feet. At what rate is the water level rising when the water is 20 feet deep?

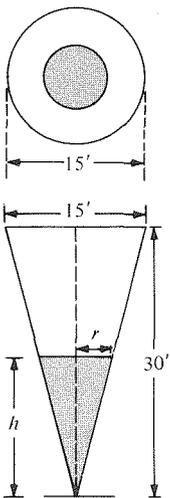


Figure 2.4.4. A conical tank partially filled with water.

Solution Figure 2.4.4 shows top and side views of the partially filled tank, both of which will be useful for our solution.

We denote by h the height of the water in the tank, so that dh/dt is the rate to be found. To proceed, we need to use the fact that the rate of rainfall is 2 inches per hour. What this means is that the water level in a cylindrical tank would rise uniformly at the rate of 2 inches per hour, so that the volume of the water pouring every hour into a circle of diameter 15 feet is $\pi \cdot (7.5)^2 \cdot \frac{1}{6}$ cubic feet $= \frac{75}{8}\pi$ cubic feet. It is useful, then, to introduce the variable V representing the volume of water in the tank; we have $dV/dt = 75\pi/8$ cubic feet per hour.

Now V and h are related by the formula for the volume of a cone: $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the “base” of the cone, in this case, the radius of the water surface. From Fig. 2.4.4, we see, using similar triangles, that $r/h = 7.5/30 = 1/4$, so $r = \frac{1}{4}h$, and hence $V = \frac{1}{48}\pi h^3$. Differentiating and using the chain rule gives $dV/dt = \frac{1}{16}\pi h^2 dh/dt$. Inserting the specific data $h = 20$ and $dV/dt = \frac{75}{8}\pi$ gives the equation $\frac{75}{8}\pi = \frac{1}{16}\pi \cdot 400 dh/dt = 25\pi dh/dt$, which we may solve for dh/dt to get $dh/dt = \frac{3}{8}$. This is in feet per hour, so the water level is rising at the rate of $4\frac{1}{2}$ inches per hour. ▲

Exercises for Section 2.4

In Exercises 1–8, assume that x and y are functions of t . Relate dx/dt and dy/dt using the given relation.

1. $x^2 - y^2 = 3$
2. $x + y = 4$
3. $x - y^2 - y^3 = 4$
4. $8x^2 + 9y^2 = 5$
5. $x + y^2 = y$
6. $\sqrt{x + 2y} = x$
7. $\sqrt{x} - \sqrt{y} = 5$
8. $(x^2 + y^2 + y^3)^{3/2} = 5$

9. Sketch the curve defined by the parametric equations $x = t^2$, $y = 1 - t$, $-\infty < t < \infty$.
10. Sketch the curve described by $x = 3t + 2$, $y = 4t - 8$, $-\infty < t < \infty$.
11. What curve do the parametric equations $x = t^2$ and $y = t^6$ describe?
12. If $x = (1 + t)^2$ and $y = (1 + t)^4$, what curve does (x, y) follow for $-\infty < t < \infty$?
13. Find the equation of the tangent line to the parametric curve $x = t^2$, $y = t^3$ at $t = 5$.
14. Find the equation of the line tangent to the parametric curve $x = t^2 + 1$, $y = 1/(t^4 + 1)$ at $t = 2$.
15. Find the equation of the tangent line to the parametric curve

$$x = \sqrt{t^4 + 6t^2 + 8t}, \quad y = \frac{t^2 + 1}{\sqrt{t} - 1}$$

at $t = 3$.

16. (a) Find the slope of the parametric curve $y = t^4 + 2t$, $x = 8t$ at $t = 1$.
(b) What relationship between x and y is satisfied by the points on this curve?
(c) Verify that $dy/dx = (dy/dt)/(dx/dt)$ for this curve.
17. Suppose that $xy = 4$. Express dy/dt in terms of dx/dt when $x = 8$ and $y = \frac{1}{2}$.
18. If $x^2 + y^2 = x^5/y$ and $dy/dt = 3$ when $x = y = \sqrt{2}$, what is dx/dt at that point?
19. Suppose that $x^2 + y^2 = t$ and that $x = 3$, $y = 4$, and $dx/dt = 7$ when $t = 25$. What is dy/dt at that moment?
20. Let x and y depend on t in such a way that $(x + y)^2 + t^2 = 2t$ and such that $x = 0$ and $y = 1$ when $t = 1$. If $dx/dt = 4$ at that moment, what is dy/dt ?
21. The radius and height of a circular cylinder are changing with time in such a way that the volume remains constant at 1 liter (= 1000 cubic centimeters). If, at a certain time, the radius is 4 centimeters and is increasing at the rate of $\frac{1}{2}$ centimeter per second, what is the rate of change of the height?
22. A hurricane is dropping 10 inches of rain per hour into a swimming pool which measures 40 feet long by 20 feet wide.
(a) What is the rate at which the volume of water in the pool is increasing?
(b) If the pool is 4 feet deep at the shallow end and 8 feet deep at the deep end, how fast is the water level rising after 2 hours? (Suppose the pool was empty to begin with.) How fast after 6 hours?
23. Water is being pumped from a 20-meter square pond into a round pond with radius 10 meters. At a certain moment, the water level in the square pond is dropping by 2 inches per minute. How fast is the water rising in the round pond?
24. A ladder 25 feet long is leaning against a vertical wall. The bottom is being shoved along the ground, towards the wall at $1\frac{1}{2}$ feet per second. How fast is the top rising when it is 15 feet off the ground?
25. A point in the plane moves in such a way that it is always twice as far from $(0, 0)$ as it is from $(0, 1)$.
(a) Show that the point moves on a circle.
(b) At the moment when the point crosses the segment between $(0, 0)$ and $(0, 1)$, what is dy/dt ?
(c) Where is the point when $dy/dt = dx/dt$? (You may assume that dx/dt and dy/dt are not simultaneously zero.)
26. Two quantities p and q depending on t are subject to the relation $1/p + 1/q = 1$.
(a) Find a relation between dp/dt and dq/dt .
(b) At a certain moment, $p = \frac{4}{3}$ and $dp/dt = 2$. What are q and dq/dt ?
27. Suppose the quantities x , y , and z are related by the equation $x^2 + y^2 + z^2 = 14$. If $dx/dt = 2$ and $dy/dt = 3$ when $x = 2$, $y = 1$, and $z = 3$, what is dz/dt ?
28. The pressure P , volume V , and temperature T of a gas are related by the law $PV/T = \text{constant}$. Find a relation between the time derivatives of P , V , and T .
29. The area of a rectangle is kept fixed at 25 square meters while the length of the sides varies. Find the rate of change of the length of one side with respect to the other when the rectangle is a square.
30. The surface area of a cube is growing at the rate of 4 square centimeters per second. How fast is the length of a side growing when the cube has sides 2 centimeters long?
- ★31. (a) Give a rule for determining when the tangent line to a parametric curve $x = f(t)$, $y = g(t)$ is horizontal and when it is vertical.
(b) When is the tangent line to the curve $x = t^2$, $y = t^3 - t$ horizontal? When is it vertical?
- ★32. (a) At which points is the tangent line to a parametric curve parallel to the line $y = x$?
(b) When is the tangent line to the curve in part (b) of Exercise 31 parallel to the line $y = x$?
(c) Sketch the curve of Exercise 31.

- ★33. Read Example 8. Show that for any conical tank, the ratio of dh/dt to the rate of rainfall is equal to the ratio between the area of the tank's open-

ing to the area of the water surface. (In fact, this result is true for a tank of any shape; see Review Exercise 32, Chapter 9.)

2.5 Antiderivatives

An antiderivative of f is a function whose derivative is f .

Many applications of calculus require one to find a function whose derivative is given. In this section, we show how to solve simple problems of this type.

Example 1 Find a function whose derivative is $2x + 3$.

Solution We recall that the derivative of x^2 is $2x$ and that the derivative of $3x$ is 3 , so the unknown function could be $x^2 + 3x$. We may check our answer by differentiating: $(d/dx)(x^2 + 3x) = 2x + 3$. ▲

The function $x^2 + 3x$ is not the only possible solution to Example 1; so are $x^2 + 3x + 1$, $x^2 + 3x + 2$, etc. In fact, since the derivative of a constant function is zero, $x^2 + 3x + C$ solves the problem for any number C .

A function F for which $F' = f$ is called an *antiderivative* of f . Unlike the derivative, the antiderivative of a function is never unique. Indeed, if F is an antiderivative of f , so is $F + C$ for an arbitrary constant C . In Section 3.6 we will show that *all* the antiderivatives are of this form. For now, we take this fact for granted. We can make the solution of an antidifferentiation problem unique by imposing an extra condition on the unknown function (see Fig. 2.5.1). The following example is a typical application of antidifferentiation.

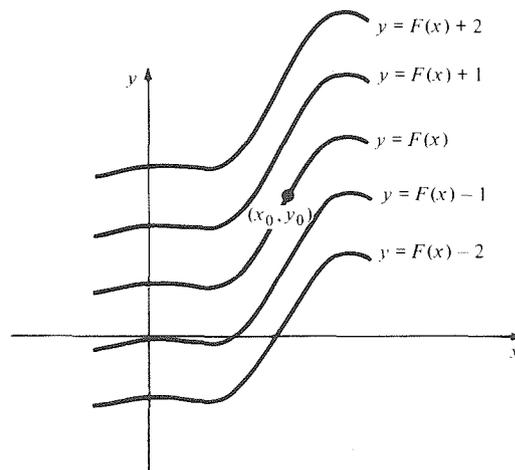


Figure 2.5.1. All these functions have the same derivative. Specifying $F(x_0) = y_0$ picks out one of them.

Example 2 The velocity of a particle moving along a line is $3t + 5$ at time t . At time 1, the particle is at position 4. Where is it at time 10?

Solution Let $F(t)$ denote the position of the particle at time t . We will determine the function F . Since velocity is the rate of change of position with respect to time, we must have $F'(t) = 3t + 5$; that is, F is an antiderivative of $f(t) = 3t + 5$. A function whose derivative is $3t$ is $\frac{3}{2}t^2$, since $(d/dt)\frac{3}{2}t^2 = \frac{3}{2}2t = 3t$. Similarly, a function whose derivative is 5 is $5t$. Therefore, we take

$$F(t) = \frac{3}{2}t^2 + 5t + C,$$

where C is a constant to be determined. To find the value of C , we use the

information that the particle is at position 4 at time 1; that is, $F(1) = 4$. Substituting 1 for t and 4 for $F(t)$ in the equation $F(t) = \frac{3}{2}t^2 + 5t + C$ gives

$$4 = \frac{3}{2} + 5 + C = \frac{13}{2} + C,$$

or $C = -\frac{5}{2}$, and so $F(t) = \frac{3}{2}t^2 + 5t - \frac{5}{2}$. Finally, we substitute 10 for t , obtaining the position at time 10: $F(10) = \frac{3}{2} \cdot 100 + 5 \cdot 10 - \frac{5}{2} = 197\frac{1}{2}$. ▲

At this point in our study of calculus, we must solve antidifferentiation problems by guessing the answer and then checking and refining our guesses if necessary. More systematic methods will be given shortly.

Example 3 Find the general antiderivative for the function $f(x) = x^4 + 6$.

Solution We may begin by looking for an antiderivative for x^4 . If we guess x^5 , the derivative is $5x^4$, which is five times too big, so we make a new guess, $\frac{1}{5}x^5$, which works. An antiderivative for 6 is $6x$. Adding our two results gives $\frac{1}{5}x^5 + 6x$; differentiating $\frac{1}{5}x^5 + 6x$ gives $x^4 + 6$, so $\frac{1}{5}x^5 + 6x$ is an antiderivative for $x^4 + 6$. We may add an arbitrary constant to get the general antiderivative $\frac{1}{5}x^5 + 6x + C$. ▲

Example 4 The acceleration of a falling body near the earth's surface is 9.8 meters per second per second. If the body has a downward velocity v_0 at $t = 0$, what is its velocity at time t ? If the position is x_0 at time 0, what is the position at time t ? (See Fig. 2.5.2.)

Solution We measure the position x in the downward direction. Let v be the velocity. Then $dv/dt = 9.8$; since an antiderivative of 9.8 is $9.8t$, we have $v = 9.8t + C$. At $t = 0$, $v = v_0$, so $v_0 = (9.8)0 + C = C$, and so $v = 9.8t + v_0$. Now $dx/dt = v = 9.8t + v_0$. Since an antiderivative of $9.8t$ is $(9.8/2)t^2 = 4.9t^2$ and an antiderivative of v_0 is v_0t , we have $x = 4.9t^2 + v_0t + D$. At $t = 0$, $x = x_0$, so $x_0 = 4.9(0)^2 + v_0 \cdot 0 + D = D$, and so $x = 4.9t^2 + v_0t + x_0$. ▲

The most commonly used notation for the antiderivative is due to Leibniz. The symbol

$$\int f(x) dx$$

denotes the class of all antiderivatives of f ; thus, if F is a particular antiderivative, we may write

$$\int f(x) dx = F(x) + C,$$

where C is an arbitrary constant. For instance, the result of Example 3 may be written

$$\int (x^4 + 6) dx = \frac{1}{5}x^5 + 6x + C.$$

The elongated S , called an *integral sign*, was introduced by Leibniz because antidifferentiation (also called *integration*) turns out to be a form of continuous summation. In Chapter 4, we will study this aspect of integration in detail. There and also in the supplement to this section, we explain the presence of the “ dx ” in the notation. For now, we simply think of dx as indicating that the independent variable is x . Its presence should also serve as a reminder that integrating is inverse to differentiating, where the dx occurs in the denominator of dy/dx .

The function $f(x)$ in $\int f(x) dx$ is called the *integrand*, and $\int f(x) dx$ is called the *indefinite integral* of $f(x)$. One traditionally refers to $f(x) dx$ as being “under” the integral sign, even though this is typographically inaccurate.

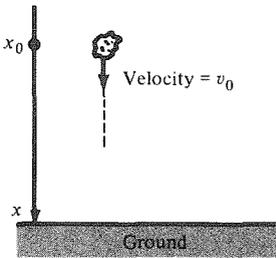


Figure 2.5.2. The body is moving downward at $t = 0$ with velocity v_0 .

Antidifferentiation and Indefinite Integrals

An *antiderivative* for f is a function F such that $F'(x) = f(x)$. We write

$$F(x) = \int f(x) dx.$$

The function $\int f(x) dx$ is also called the *indefinite integral* of f , and f is called the *integrand*.

If $F(x)$ is an antiderivative of $f(x)$, the general antiderivative has the form $F(x) + C$ for an arbitrary constant C .

Some of the differentiation rules lead directly to systematic rules for antidifferentiation. The rules in the following box can be proved by differentiation of their right-hand sides (see Example 5 below).

Antidifferentiation Rules

Sum:
$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Constant multiple:
$$\int af(x) dx = a \int f(x) dx, \quad \text{where } a \text{ is constant}$$

Power:
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

Polynomial:
$$\int (c_n x^n + \cdots + c_1 x + c_0) dx = \frac{c_n}{n+1} x^{n+1} + \cdots + \frac{c_1}{2} x^2 + c_0 x + C$$

Example 5 Prove the power rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

Solution By definition, $F(x) = \int x^n dx$ is a function such that $F'(x) = f(x) = x^n$. However, $F(x) = [x^{n+1}/(n+1)] + C$ is such a function since its derivative is $F'(x) = (n+1) \cdot x^{n+1-1}/(n+1) = x^n$, by the power rule for derivatives. ▲

The exclusion $n \neq -1$ in the power rule arises because the formula $x^{n+1}/(n+1)$ makes no sense for $n = -1$; the denominator is zero. (It turns out that $1/x = x^{-1}$ does have an antiderivative, but it is a logarithm function rather than a power of x . We will study logarithms in Chapter 6.)

Example 6 Find $\int \left[\frac{1}{x^2} + 3x + 2 - \frac{8}{\sqrt{x}} \right] dx$.

Solution

$$\begin{aligned} \int \left(\frac{1}{x^2} + 3x + 2 - \frac{8}{\sqrt{x}} \right) dx \\ &= \int \frac{1}{x^2} dx + 3 \int x dx + 2 \int 1 dx - 8 \int \frac{1}{\sqrt{x}} dx \\ &= \int x^{-2} dx + 3 \int x^1 dx + 2 \int x^0 dx - 8 \int x^{-1/2} dx \end{aligned}$$

$$\begin{aligned}
 &= -1x^{-1} + \frac{3}{2}x^2 + 2x^1 - \frac{8}{1/2}x^{1/2} + C \\
 &= -\frac{1}{x} + \frac{3}{2}x^2 + 2x - 16\sqrt{x} + C.
 \end{aligned}$$

We write C only once because the sum of four constants is a constant. \blacktriangle

Example 7 Find $\int \frac{dx}{(3x+1)^5}$.

Solution We are looking for an antiderivative of $1/(3x+1)^5$. The power of a function rule suggests that we guess $1/(3x+1)^4$. Differentiating, we have

$$\frac{d}{dx} \frac{1}{(3x+1)^4} = \frac{-4}{(3x+1)^5} \frac{d}{dx} (3x+1) = \frac{-12}{(3x+1)^5}$$

Comparing with $1/(3x+1)^5$, we see that we are off by a factor of -12 , so

$$\int \frac{dx}{(3x+1)^5} = \frac{-1}{12(3x+1)^4} + C. \blacktriangle$$

Using the same method as in Example 7, we find that

$$\int (ax+b)^n dx = \frac{1}{a(n+1)} (ax+b)^{n+1} + C,$$

where a and b are constants, $a \neq 0$, and n is a rational number, $n \neq -1$.

Example 8 Find $\int \sqrt{3x+2} dx$.

Solution By the formula for $\int (ax+b)^n dx$ with $a=3$, $b=2$, and $n=\frac{1}{2}$, we get

$$\begin{aligned}
 \int \sqrt{3x+2} dx &= \int (3x+2)^{1/2} dx \\
 &= \frac{1}{3(3/2)} (3x+2)^{3/2} + C = \frac{2}{9} (3x+2)^{3/2} + C. \blacktriangle
 \end{aligned}$$

Example 9 Find $\int \frac{x^3-8}{x-2} dx$.

Solution Here we simplify first. Dividing x^3-8 by $x-2$ gives $(x^3-8)/(x-2) = x^2+2x+4$. Thus

$$\int \frac{x^3-8}{x-2} dx = \frac{x^3}{3} + x^2 + 4x + C. \blacktriangle$$

Example 10 Let x = position, v = velocity, a = acceleration, t = time. Express the relations between these variables by using the indefinite integral notation.

Solution By the definitions of velocity and acceleration, we have $v = dx/dt$ and $a = dv/dt$. It follows that

$$v = \int a dt \quad \text{and} \quad x = \int v dt. \blacktriangle$$

Example 11 Water is flowing into a tub at $3t + 1/(t+1)^2$ gallons per minute after t minutes. How much water is in the tub after 2 minutes if it started out empty?

Solution Let $f(t)$ be the amount of water (in gallons) in the tub at time t . We are given

$$f'(t) = 3t + \frac{1}{(t+1)^2}.$$

This equation means that f is the antiderivative of $3t + 1/(t+1)^2$; thus,

$$\begin{aligned} f(t) &= \int \left(3t + \frac{1}{(t+1)^2} \right) dt \\ &= \int 3t dt + \int \frac{1}{(t+1)^2} dt = \frac{3t^2}{2} - \frac{1}{t+1} + C. \end{aligned}$$

Since the tub started out empty, $f(0) = 0$; so $0 = -1 + C$, and thus $C = 1$. Therefore $f(t) = 3t^2/2 - 1/(t+1) + 1$. Setting $t = 2$ gives $f(2) = 3 \cdot 4/2 - 1/3 + 1 = 6\frac{2}{3}$ gallons after 2 minutes. \blacktriangle

Supplement to Section 2.5

The Notation $\int f(x) dx$

The Leibniz notation $\int f(x) dx$ for the antiderivative of a function $f(x)$ may seem strange at this point, but it is really rather natural and remarkably functional. To motivate it, let us study the velocity–distance relationship again. As in Section 1.1, we imagine a bus moving on a straight road with position $y = F(x)$ in meters from a designated starting point at time x in seconds (see Fig. 1.1.1). There we showed that $v = dy/dx$ is the velocity of the bus. As in Section 1.3, we may motivate this notation by writing the velocity as the limit

$$\frac{\Delta y}{\Delta x} = \frac{\text{distance travelled}}{\text{elapsed time}} \quad \text{as } \Delta x \rightarrow 0.$$

Conversely, to reconstruct y from a given velocity function $v = f(x)$, we notice that in a short time interval from x to $x + \Delta x$, the bus has gone approximately $\Delta y \approx f(x) \Delta x$ meters (distance travelled = velocity \times time elapsed). The total distance travelled is thus the sum of $f(x) \Delta x$ over all the little Δx 's making up the total time of the trip. This abbreviates to $\int f(x) dx$.

On the other hand, the distance travelled is $y = F(x)$, assuming $F(0) = 0$, and we know that $dy/dx = v = f(x)$, i.e., F is an antiderivative of f . Thus $\int f(x) dx$ is a reasonable notation for this antiderivative. The arbitrariness in the starting position $F(0)$ corresponds to the arbitrary constant that can be added to the antiderivative.

Exercises for Section 2.5

Find antiderivatives for each of the functions in Exercises 1–8.

- | | |
|-------------------------|---|
| 1. $x + 2$ | 2. $x^6 + 9$ |
| 3. $s(s+1)(s+2)$ | 4. $4x^8 + 3x^2$ |
| 5. $\frac{1}{t^3}$ | 6. $x^5 + \frac{2}{x^4}$ |
| 7. $x^{3/2} - \sqrt{x}$ | 8. $x^4 - \frac{1}{\sqrt{x}} + x^{3/2}$ |

In Exercises 9–12, v is the velocity of a particle on the line, and $F(t)$ is the position at time t .

9. $v = 8t + 2$; $F(0) = 0$; find $F(1)$.
10. $v = -2t + 3$; $F(1) = 2$; find $F(3)$.

11. $v = t^2 + \sqrt{t}$; $F(1) = 1$; find $F(\frac{1}{2})$.

12. $v = t^{3/2} - t^2$; $F(2) = 1$; find $F(1)$.

Find the general antiderivatives for the functions f given in Exercises 13–20.

13. $f(x) = 3x$

14. $f(x) = 3x^4 + 4x^3$

15. $f(x) = \frac{x+1}{x^3}$

16. $f(t) = (t+1)^2$

17. $f(x) = \sqrt{x+1}$

18. $f(x) = (\sqrt{x} + 1)^2$

19. $f(t) = (t+1)^{3/2}$

20. $f(s) = (s+8)^{5/8}$

In Exercises 21–24, the velocity v_0 of a falling body (in meters per second) near the earth's surface is given at time $t = 0$. Find the velocity at time t and the position

at time t with the given initial positions x_0 . (The x axis is oriented downwards as in Fig. 2.5.2.)

- 21. $v_0 = 1; x_0 = 2$
- 22. $v_0 = 3; x_0 = -1$
- 23. $v_0 = -2; x_0 = 0$
- 24. $v_0 = -2; x_0 = -6$
- 25. Is it true that $\int f(x)g(x)dx$ is equal to $[\int f(x)dx]g(x) + f(x)[\int g(x)dx]$?
- 26. Prove the constant multiple rule for antidifferentiation.
- 27. Prove the sum rule for antiderivatives.
- 28. Prove that $\int f(x)f'(x)dx = \frac{1}{2}[f(x)]^2 + C$ for any function f .

Find the indefinite integrals in Exercises 29–40.

- 29. $\int (x^2 + 3x + 2)dx$
- 30. $\int 4\pi r^2 dr$
- 31. $\int (3t^2 + 2t + 1)dt$
- 32. $\int (u^4 - 6u)du$
- 33. $\int (8t + 1)^{-2}dt$
- 34. $\int \left(\frac{t^3 + t + 1}{t^5}\right)dt$
- 35. $\int \frac{4}{(3b + 2)^9}db$
- 36. $\int \sqrt{2u + 5}du$
- 37. $\int \left(\frac{1}{x^4} + x^4\right)dx$
- 38. $\int \left(\frac{t^2 - t + 2}{t^6}\right)dt$
- 39. $\int \frac{x^2 + 3}{\sqrt{x}}dx$
- 40. $\int \frac{t^3 - 8t + 1}{t^{3/2}}dt$

Find the indicated antiderivatives in Exercises 41–52.

- 41. $\int (x^3 + 3x)dx$
- 42. $\int (t^3 + t^{-2})dt$
- 43. $\int \frac{1}{(t + 1)^2}dt$
- 44. $\int \frac{w^2 + 2}{w^5}dw$
- 45. $\int \sqrt{8x + 3}dx$
- 46. $\int \sqrt{10x - 3}dx$
- 47. $\int 10(8 - 3x)^{3/2}dx$
- 48. $\int 3(x - 1)^{5/2}dx$
- 49. $\int \frac{\sqrt{x - 1} + 3}{(x - 1)^{1/2}}dx$
- 50. $\int \frac{(3x - 2)^{3/2} - 8}{\sqrt{3x - 2}}dx$
- 51. $\int \frac{x^3 - 27}{x - 3}dx$
- 52. $\int \frac{x^4 - 1}{x - 1}dx$

- 53. A ball is thrown downward with a velocity of 10 meters per second. How long does it take the ball to fall 150 meters?
- 54. A particle moves along a line with velocity $v(t) = \frac{1}{2}t^2 + t$. If it is at $x = 0$ when $t = 0$, find its position as a function of t .
- 55. The population of Booneville increases at a rate of $r(t) = (3.62)(1 + 0.8t^2)$ people per year, where t is the time in years from 1970. The population in 1976 was 726. What was it in 1984?
- 56. A car accelerates from rest to 55 miles per hour in 12 seconds. Assuming that the acceleration is constant, how far does the car travel during those 12 seconds?
- 57. A rock is thrown vertically upward with velocity 19.6 meters per second. After how long does it return to the thrower? (The acceleration due to gravity is 9.8 meters per second per second; see Example 4 and Fig. 2.5.3.)

- 58. Suppose that the marginal cost of producing grumbies at production level p is $100/(p + 20)^2$. If the cost of production is 100 when $p = 0$ (setup costs), what is the cost when $p = 80$?

- 59. (a) Find $\frac{d}{dx} \left(\frac{x^3 + 1}{x^3 - 1} \right)$.
- (b) Find $\int \frac{-6x^2}{(x^3 - 1)^2} dx$.
- 60. (a) Find $(d/dx)\sqrt{x^2 + 2x}$.
- (b) Find $\int \frac{z + 1}{\sqrt{z^2 + 2z}} dz$.
- 61. (a) Differentiate $(x^4 + 1)^{20}$.
- (b) Find $\int [(x^4 + 1)^{19}x^3 + 3x^{2/3}] dx$.
- 62. (a) Differentiate $\frac{1}{3 + x^{9/2}}$.
- (b) Find $\int \frac{x^{7/2}}{(3 + x^{9/2})^2} dx$.
- 63. Find a function $F(x)$ such that $x^5F'(x) + x^3 + 2x = 3$.
- 64. Find a function $f(x)$ whose graph passes through $(1, 1)$ and such that the slope of its tangent line at $(x, f(x))$ is $3x + 1$.
- 65. Find the antiderivative $F(x)$ of the function $f(x) = x^3 + 3x^2 + 2$ which satisfies $F(0) = 1$.
- 66. Find the antiderivative $G(y)$ of $g(y) = (y + 4)^2$ which satisfies $G(1) = 0$.
- *67. (a) What integration formula can you derive from the general power of a function rule? (See Exercises 60 and 61.) (b) Find $\int (x^3 + 4)^{-3}x^2 dx$.
- *68. (a) What integration formula can you derive from the chain rule?
- (b) Find $\int [\sqrt{x^2 + 20x} + (x^2 + 20x)](2x + 20) dx$.

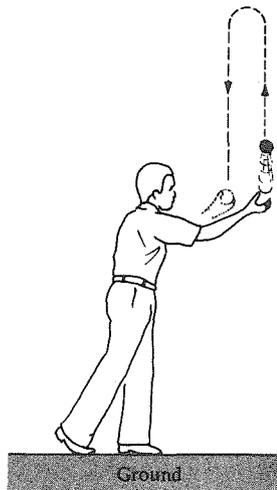


Figure 2.5.3. The path of a rock thrown upwards from the earth.

Review Exercises for Chapter 2

Differentiate each of the functions in Exercises 1–10.

1. $(6x + 1)^3$
2. $(x^2 + 9x + 10)^8$
3. $(x^3 + x^2 - 1)^{10}$
4. $(x^2 + 1)^{-13}$
5. $6/x$
6. $\frac{9x^9 - x^8 + 14x^7 + x^6 + 5x^4 + x^2 + 2}{x^2}$
7. $\frac{x + 1}{x - 1}$
8. $\frac{x^2 - 1}{x^2 - 2}$
9. $\frac{(x^2 + 1)^{13}}{(x^2 - 1)^{14}}$
10. $\frac{[(x^3 + 6)^2 - (2x^4 + 1)^3]}{(x^5 + 8)}$

In Exercises 11–20, let

$$A(x) = x^3 - x^2 - 2x,$$

$$B(x) = x^2 + \frac{1}{4}x - \frac{3}{8},$$

$$C(x) = 2x^3 - 5x^2 + x + 2,$$

$$D(x) = x^2 + 8x + 16.$$

Differentiate the given functions in Exercises 11–16.

11. $[B(x)]^3$
12. $[A(x)]^2$
13. $\frac{A(x)}{D(x)}$
14. $C(x) - 3\left[\frac{D(x)}{A(x)}\right]$
15. $A(2x)$
16. $A(2B(2x))$

Find the equation of the tangent line to the graph of the given function in Exercises 17–20, where A , B , C , and D are given above.

17. $[A(x)]^{1/3}$ at $x = 1$
18. $[B(x)]^2$ at $x = 0$
19. $[C(x)]^2$ at $x = -2$
20. $\sqrt{D(x)}$ at $x = -1$

Differentiate the functions in Exercises 21–28.

21. $f(x) = x^{5/3}$
22. $h(x) = (1 + 2x^{1/2})^{3/2}$
23. $g(x) = \frac{x^{3/2}}{\sqrt{1 + x^2}}$
24. $l(y) = \frac{y^2}{\sqrt{1 - y^3}}$
25. $f(x) = \frac{1 + x^{3/2}}{1 - x^{3/2}}$
26. $f(y) = y^3 + \left(\frac{1 + y^3}{1 - y^3}\right)^{1/2}$
27. $f(x) = \frac{8\sqrt{x}}{1 + \sqrt{x}} + 3x\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right)$
28. $f(y) = \frac{8y^4}{1 + [3/(1 + \sqrt{y})]}$

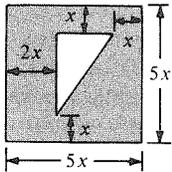
Find the first and second derivatives of the functions in Exercises 29–40.

29. $f(x) = \frac{x - a}{x^2 + 2bx + c}$ (a, b, c constants).
30. $f(z) = \frac{az + b}{cz + d}$ (a, b, c, d constants).
31. $x(t) = \left(\frac{A}{1 - t}\right) + \left(\frac{B}{1 - t^2}\right) + \left(\frac{C}{1 - t^3}\right)$
(A, B, C constants).
32. $s(t) = (t - 1)(t + 1)$.
33. $h(r) = r^{13} - \sqrt{2}r^4 - \left(\frac{r}{r^2 + 3}\right)$
34. $k(s) = s^{15} + \left(\frac{s^{12} - 2}{s - 1}\right) + s^2 - 1$.
35. $f(x) = (x - 1)^3g(x)$ (here $g(x)$ is some differentiable function).
36. $V(r) = \frac{4}{3}\pi r^2 + 2\pi rh(r)$, where $h(r) = 2r - 1$.
37. $h(x) = (x - 2)^4(x^2 + 2)$
38. $f(x) = 1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^{10}}$
39. $g(t) = \frac{t^5 - 3t^4}{t^3 - 1}$
40. $q(s) = s^5\left(\frac{1 - 2s}{1 - s}\right)$
41. The volume of a falling spherical raindrop grows at a rate which is proportional to the surface area of the drop. Show that the radius of the drop increases at a constant rate.
42. The temperature of the atmosphere decreases with altitude at the rate of 2°C per kilometer at the top of a certain cliff. A hang glider pilot finds that the outside temperature is rising at the rate of 10^{-4} degrees Centigrade per second. How fast is the glider falling?
43. For temperatures in the range $[-50, 150]$ (degrees Celsius), the pressure in a certain closed container of gas changes linearly with the temperature. Suppose that a 40° increase in temperature causes the pressure to increase by 30 millibars (a millibar is one thousandth of the average atmospheric pressure at sea level). (a) What is the rate of change of pressure with respect to temperature? (b) What change of temperature would cause the pressure to drop by 9 millibars?
44. Find the rate of change of the length of an edge of a cube with respect to its surface area.
45. The organism *amoebus rectilineus* always maintains the shape of a right triangle whose area is 10^{-6} square millimeters. Find the rate of change of the perimeter at a moment when the organism is isosceles and one of the legs is growing at 10^{-4} millimeters per second.

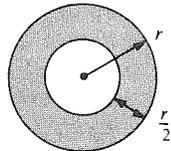
46. The price of calculus books rises at the rate of 75¢ per year. The price of books varies with weight at a rate of \$2.00 per pound. How fast is the weight of books rising? (Ignore the effect of inflation).
47. Two ships, A and B , leave San Francisco together and sail due west. A sails at 20 miles per hour and B at 25 miles per hour. Ten miles out to sea, A turns due north and B continues due west. How fast are they moving away from each other 4 hours after departing San Francisco?
48. At an altitude of 2000 meters, a parachutist jumps from an airplane and falls $4.9t^2$ meters in t seconds. Suppose that the air pressure p decreases with altitude at the constant rate of 0.095 gsc per meter. The parachutist's ears pop when dp/dt reaches 2 gsc per second. At what time does this happen?

In Exercises 49–52, let A represent the area of the shaded region in Fig. 2.R.1.

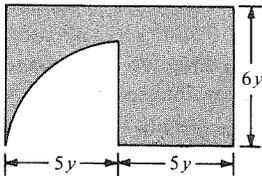
49. Find dA/dx and d^2A/dx^2 .
 50. Find dA/dr and d^2A/dr^2 .
 51. Find dA/dy and d^2A/dy^2 .
 52. Find dA/dx and d^2A/dx^2 .



49 Find dA/dx and d^2A/dx^2

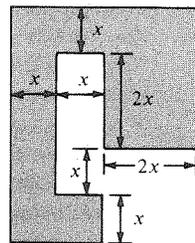


50 Find dA/dr and d^2A/dr^2



(Rectangle with $\frac{1}{4}$ of disk removed)

51 Find dA/dy and d^2A/dy^2



52 Find dA/dx and d^2A/dx^2

Figure 2.R.1. Find the indicated rates of change of the areas.

In Exercises 53–56 let P represent the perimeter of the shaded region in Figure 2.R.1.

53. For Exercise 49, find dA/dP and dP/dx .
 54. For Exercise 50, find dA/dP and dP/dr .
 55. For Exercise 51, find dA/dP and dP/dy .
 56. For Exercise 52, find dA/dP and dP/dx .
 57. The total cost C in dollars for producing x cases of solvent is given by $C(x) = 20 + 5x - (0.01)x^2$.

The number 20 in the formula represents the fixed cost for placing the order, regardless of size. The other terms represent the variable costs.

- (a) Find the marginal cost.
 (b) Find the cost for the 85th case of solvent, i.e., the marginal cost for a purchase of 84 cases.
 (c) Explain in the language of marginal cost the statement “the more you buy, the cheaper it gets.”
 (d) Find a large value of x , beyond which it is unreasonable for the given formula for $C(x)$ to be applicable.
58. In Exercise 57, suppose that the solvent is priced at $8 - [(8x + 100)/(x + 300)]$ dollars per case at production level x . Calculate the marginal revenue and the marginal profit.

Find the equation of the line tangent to the graph of the function at the indicated point in Exercises 59–62.

59. $f(x) = (x^3 - 6x)^3$; $(0, 0)$

60. $f(x) = \frac{x^4 - 1}{6x^2 + 1}$; $(1, 0)$

61. $f(x) = \frac{x^3 - 7}{x^3 + 11}$; $(2, \frac{1}{19})$

62. $f(x) = \frac{x^5 - 6x^4 + 2x^3 - x}{x^2 + 1}$; $(1, -2)$

63. If $x^2 + y^2 + xy^3 = 1$, find dy/dx when $x = 0$, $y = 1$.

64. If x and y are functions of t , $x^4 + xy + y^4 = 2$, and $dy/dt = 1$ at $x = 1$, $y = 1$, find dx/dt at $x = 1$, $y = 1$.

65. Let a curve be described by the parametric equations

$$x = \sqrt{t} + t^2 + \frac{1}{t}, \quad 1 \leq t \leq 3.$$

$$y = 1 + \sqrt[3]{t} + t,$$

Find the equation of the tangent line at $t = 2$.

66. The speed of an object traveling on a parametric curve is given by $v = \sqrt{(dx/dt)^2 + (dy/dt)^2}$.

(a) Find the speed at $t = 1$ for the motion $x = t^3 - 3t^2 + 1$, $y = t^5 - t^7$.

(b) Repeat for $x = t^2 - 3$, $y = \frac{1}{3}t^3 - t$ at $t = 1$.

67. Find the linear approximations for: (a) $\sqrt[3]{27.11}$; (b) $\sqrt[3]{-63.01}$.

68. Find the linear approximations for (a) $\sqrt[5]{32.02}$ and (b) $\sqrt[6]{63.98}$.

69. (a) Find the linear approximation to the function $(x^{40} - 1)/(x^{29} + 1)$ at $x_0 = 1$.

(b) Calculate $[(1.021)^{40} - 1]/[(1.021)^{29} + 1]$ approximately.

70. Find an approximate value for $\sqrt{1 + (0.0036)^2}$.

71. Find a formula for $(d^2/dx^2)[f(x)g(x)]$.

72. If f is a given differentiable function and $g(x) = f(\sqrt{x})$, what is $g'(x)$?

73. Differentiate both sides of the equation

$$[f(x)^m]^n = f(x)^{mn}$$

where m and n are positive integers and show that you get the same result on each side.

74. Find a formula for $(d/dx)[f(x)^m g(x)^n]$, where f and g are differentiable functions and m and n are positive integers.

Find the antiderivatives in Exercises 75–94.

75. $\int 10 \, dx$

76. $\int (4.9t + 15) \, dt$

77. $\int (4x^3 + 3x^2 + 2x + 1) \, dx$

78. $\int (s^5 + 4s^4 + 9) \, ds$

79. $\int \frac{2}{3} x^{2/5} \, dx$

80. $\int 4x^{3/2} \, dx$

81. $\int \left[\frac{-1}{x^2} + \frac{-2}{x^3} + \frac{-3}{x^4} + \frac{-4}{x^5} \right] \, dx$

82. $\int \left(\frac{1}{z^2} + \frac{4}{z^3} \right) \, dz$

83. $\int (x^2 + \sqrt{x}) \, dx$

84. $\int (x + \sqrt{x} + \sqrt[3]{x}) \, dx$

85. $\int (x^{3/2} + x^{-1/2}) \, dx$

86. $\int \left(\frac{2}{\sqrt{x}} \right) \, dx$

87. $\int \frac{x^4 + x^6 + 1}{x^{1/2}} \, dx$

88. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) \, dx$

89. $\int \sqrt{x-1} \, dx$

90. $\int (2x + 1)^{3/2} \, dx$

91. $\int \frac{1}{(x-1)^2} \, dx$

92. $\int [(8x-10)^{3/2} + 10x] \, dx$

93. $\int [(x-1)^{1/2} - (x-2)^{5/2}] \, dx$

94. $\int \frac{(2x-1)^{3/2} - 1}{(10x-5)^{1/2}} \, dx$

Differentiate each of the functions in Exercises 95–102 and write the corresponding antiderivation formula.

95. $f(x) = \frac{x^{1/3}}{(x^{1/2} - x^{1/3} + 1)^{1/2}}$

96. $f(x) = \sqrt{\left(\frac{8x^2 + 8x + 1}{x^{1/2} + x^{-1/2}} \right)}$

97. $f(x) = \sqrt{x} - \sqrt{\frac{x-1}{x+1}}$

98. $f(x) = \sqrt{\frac{\sqrt{x}-1}{\sqrt{x}+1}}$

99. $f(x) = \frac{\sqrt[4]{x}-1}{\sqrt[4]{x}+1}$

100. $f(x) = \sqrt[3]{\frac{x-1}{3x+1}}$

101. $f(x) = \sqrt{\frac{x^2+1}{x^2-1}}$

102. $f(x) = \sqrt{x^3 + 2\sqrt{x} + 1}$

103. The *lemniscate* $3(x^2 + y^2)^2 = 25(x^2 - y^2)$ is a planar curve which intersects itself at the origin.

(a) Show by use of symmetry that the entire lemniscate can be graphed by (1) reflecting the first quadrant portion through the x axis, and then (2) reflecting the right half-plane portion through the origin to the left half-plane.

(b) Find by means of implicit differentiation the value of dy/dx at $(2, 1)$.

(c) Determine the equation of the tangent line to the lemniscate through $(2, 1)$.

104. The *drag* on an automobile is the force opposing its motion down the highway, due largely to air resistance. The drag in pounds D can be approximated for velocities v near 50 miles per hour by $D = kv^2$. Using $k = 0.24$, find the rate of increase of drag with respect to time at 55 miles per hour when the automobile is undergoing an acceleration of 3 miles per hour each second.

105. The air resistance of an aircraft fuel tank is given approximately by $D = 980 + 7(v - 700)$ lbs for the velocity range of $700 \leq v \leq 800$ miles per hour. Find the rate of increase in air resistance with respect to time as the aircraft accelerates past the speed of sound (740 miles per hour) at a constant rate of 12 miles per hour each second.

106. A physiology experiment measures the heart rate $R(x)$ in beats per minute of an athlete climbing a vertical rope of length x feet. The experiment produces two graphs: one is the heart rate R versus the length x ; the other is the length x versus the time t in seconds it took to climb the rope (from a fresh start, as fast as possible).

(a) Give a formula for the change in heart rate in going from a 12-second climb to a 13-second climb using the linear approximation.

(b) Explain how to use the tangent lines to the two graphs and the chain rule to compute the change in part (a).

★107. (a) Find a formula for the second and third derivatives of x^n .

(b) Find a formula for the r th derivative of x^n if $n > r$.

(c) Find a formula for the derivative of the product $f(x)g(x)h(x)$ of three functions.

- ★108. (a) Prove that if f/g is a rational function (i.e., a quotient of polynomials) with derivative zero, then f/g is a constant.
(b) Conclude that if the rational functions F

- and G are both antiderivatives for a function h , then F and G differ by a constant.
★109. Prove that if the k th derivative of a rational function $r(x)$ is zero for some k , then $r(x)$ is a polynomial.