ELEMENTARY MATHEMATICS

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Chapter 1

BASIC ALGEBRA

1.1. The Real Number System

We shall assume that the reader has a knowledge of the real numbers. Some examples of real numbers are 3, $1/2$, π, $\sqrt{23}$ and $-\sqrt[3]{5}$. We shall denote the collection of all real numbers by $\mathbb{R}$, and write $x \in \mathbb{R}$ to denote that x is a real number.

Among the real numbers are the collection $\mathbb{N}$ of all natural numbers and the collection $\mathbb{Z}$ of all integers. These are given by

$$\mathbb{N} = \{1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Another subcollection of the real numbers is the collection $\mathbb{Q}$ of all rational numbers. To put it simply, this is the collection of all fractions. Clearly we can write any fraction if we allow the numerator to be any integer (positive, negative or zero) and insist that the denominator must be a positive integer. Hence we have

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\}.$$

It can be shown that the collection $\mathbb{Q}$ contains precisely those numbers which have terminating or repeating decimals in their decimal notations. For example,

$$\frac{3}{4} = 0.75 \qquad \text{and} \qquad -\frac{18}{13} = -1.\overline{384615}$$

are rational numbers. Here the digits with the overline repeat.

It can be shown that the number $\sqrt{2}$ is not a rational number. This is an example of a real number which is not a rational number. Indeed, any real number which is not a rational number is called an irrational number. It can be shown that any irrational number, when expressed in decimal notation, has

† This chapter was written at Macquarie University in 1999.

non-terminating and non-repeating decimals. The number π is an example of an irrational number. It is known that

$$\pi = 3.14159\ldots.$$

The digits do not terminate or repeat. In fact, a book was published some years ago giving the value of the number π to many digits and nothing else – a very uninteresting book indeed! On the other extreme, one of the states in the USA has a law which decrees that $\pi = 3$, no doubt causing a lot of problems for those who have to measure the size of your block of land.

1.2. Arithmetic

In mathematics, we often have to perform some or all of the four major operations of arithmetic on real numbers. These are addition (+), subtraction (−), multiplication (×) and division (÷). There are simple rules and conventions which we need to observe.

SOME RULES OF ARITHMETIC.

(a) *Operations within brackets are performed first.*
(b) *If there are no brackets to indicate priority, then multiplication and division take precedence over addition and subtraction.*
(c) *Addition and subtraction are performed in their order of appearance.*
(d) *Multiplication and division are performed in their order of appearance.*
(e) *A number of additions can be performed in any order. For any real numbers $a, b, c \in \mathbb{R}$, we have $a + (b + c) = (a + b) + c$ and $a + b = b + a$.*
(f) *A number of multiplications can be performed in any order. For any real numbers $a, b, c \in \mathbb{R}$, we have $a \times (b \times c) = (a \times b) \times c$ and $a \times b = b \times a$.*

Example 1.2.1. We have $-3 \times 4 - 5 + (-3) = -(3 \times 4) - 5 + (-3) = -12 - 5 - 3 = -20$. Note that we have recognized that 3×4 takes precedence over the $-$ signs.

Example 1.2.2. We have

$$\begin{aligned} 21 + 32 \div (-4) + (-6) &= 21 + (32 \div (-4)) + (-6) = 21 + (-8) + (-6) \\ &= 21 - 8 - 6 = (21 - 8) - 6 = 13 - 6 = 7. \end{aligned}$$

Note that we have recognized that $32 \div (-4)$ takes precedence over the $+$ signs, and that $21 - 8$ takes precedence over the following $-$ sign.

Example 1.2.3. We have $(366 \div (-6) - (-6)) \div (-11) = ((-61) - (-6)) \div (-11) = (-55) \div (-11) = 5$. Note that the division by -11 is performed last because of brackets.

Example 1.2.4. We have $720 \div (-9) \div 4 \times (-2) = (-80) \div 4 \times (-2) = (-20) \times (-2) = 40$.

Example 1.2.5. Convince yourself that $(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6 = 4$.

Another operation on real numbers that we perform frequently is taking square roots. Here we need to exercise great care.

Definition. Suppose that $a \geq 0$. We say that x is a square root of a if $x^2 = a$.

Remarks. (1) If $a > 0$, then there are two square roots of a. We denote by $\sqrt{a}$ the positive square root of a, and by $-\sqrt{a}$ the negative square root of a.

(2) If $a = 0$, then there is only one square root of a. We have $\sqrt{0} = 0$.

(3) Note that square root of a is not defined when $a < 0$. If x is a real number, then $x^2 \geq 0$ and so cannot be equal to any real negative number a.

EXAMPLE 1.2.6. We have $\sqrt{(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6} = 2$.

EXAMPLE 1.2.7. We have $\sqrt{27} = 3 \times \sqrt{3}$. To see this, note that

$$(3 \times \sqrt{3})^2 = 3 \times \sqrt{3} \times 3 \times \sqrt{3} = 3 \times 3 \times \sqrt{3} \times \sqrt{3} = 3 \times 3 \times 3 = 27.$$

EXAMPLE 1.2.8. We have $\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3} = 2 \times 3 \times \sqrt{2} = 6 \times \sqrt{2}$.

1.3. Distributive Laws

We now consider the distribution of multiplication inside brackets. For convenience, we usually suppress the multiplication sign $\times$, and write ab to denote the product $a \times b$.

DISTRIBUTIVE LAWS. *For every $a, b, c, d \in \mathbb{R}$, we have*
(a) $a(b + c) = ab + ac$;
(b) $(a + b)c = ac + bc$; *and*
(c) $(a + b)(c + d) = ac + ad + bc + bd$.

Special cases of part (c) above include the following two laws.

LAWS ON SQUARES. *For every $a, b \in \mathbb{R}$, we have*
(a) $(a + b)^2 = a^2 + 2ab + b^2$;
(b) $(a - b)^2 = a^2 - 2ab + b^2$; *and*
(c) $a^2 - b^2 = (a - b)(a + b)$.

PROOF. We have

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

and

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ba + b^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.$$

Also

$$(a - b)(a + b) = a^2 + ab - ba - b^2 = a^2 + ab - ab - b^2 = a^2 - b^2. \qquad \clubsuit$$

LAWS ON CUBES. *For every $a, b \in \mathbb{R}$, we have*
(a) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$; *and*
(b) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

PROOF. We have

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - ba^2 - bab - b^3 \\ &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3\end{aligned}$$

and

$$\begin{aligned}(a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 + ba^2 - bab + b^3 \\ &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3.\end{aligned} \qquad \clubsuit$$

EXAMPLE 1.3.1. Consider the expression $(2x + 5)^2 - (x + 5)^2$. Using part (a) on the Laws on squares, we have $(2x + 5)^2 = 4x^2 + 20x + 25$ and $(x + 5)^2 = x^2 + 10x + 25$. It follows that

$$\begin{aligned}(2x + 5)^2 - (x + 5)^2 &= (4x^2 + 20x + 25) - (x^2 + 10x + 25) \\ &= 4x^2 + 20x + 25 - x^2 - 10x - 25 = 3x^2 + 10x.\end{aligned}$$

EXAMPLE 1.3.2. Consider the expression $(x-y)(x+y-2)+2x$. Using an extended version of part (c) of the Distributive laws, we have

$$(x-y)(x+y-2) = x^2+xy-2x-xy-y^2+2y = x^2-2x-y^2+2y.$$

It follows that

$$(x-y)(x+y-2)+2x = (x^2-2x-y^2+2y)+2x = x^2-2x-y^2+2y+2x = x^2-y^2+2y.$$

Alternatively, we have

$$\begin{aligned}(x-y)(x+y-2)+2x &= (x-y)((x+y)-2)+2x = (x-y)(x+y)-2(x-y)+2x \\ &= x^2-y^2-(2x-2y)+2x = x^2-y^2-2x+2y+2x = x^2-y^2+2y.\end{aligned}$$

EXAMPLE 1.3.3. We have

$$\begin{aligned}(x+1)(x-2)(x+3) &= (x^2-2x+x-2)(x+3) = (x^2-x-2)(x+3) \\ &= x^3+3x^2-x^2-3x-2x-6 = x^3+2x^2-5x-6.\end{aligned}$$

EXAMPLE 1.3.4. We have

$$\begin{aligned}&(5x+3)^2-(2x-3)^2+(3x-2)(3x+2) \\ &\quad= (25x^2+30x+9)-(4x^2-12x+9)+(9x^2-4) \\ &\quad= 25x^2+30x+9-4x^2+12x-9+9x^2-4 \\ &\quad= 30x^2+42x-4.\end{aligned}$$

1.4. Arithmetic of Fractions

In this section, we discuss briefly the arithmetic of fractions. Suppose that we wish to add two fractions and consider

$$\frac{a}{b}+\frac{c}{d},$$

where $a,b,c,d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$. For convenience, we have relaxed the requirement that b and d are positive integers.

ADDITION AND SUBTRACTION OF FRACTIONS. *We have*

$$\frac{a}{b}+\frac{c}{d} = \frac{ad}{bd}+\frac{bc}{bd} = \frac{ad+bc}{bd} \qquad \text{and} \qquad \frac{a}{b}-\frac{c}{d} = \frac{ad}{bd}-\frac{bc}{bd} = \frac{ad-bc}{bd}.$$

In both cases, we first rewrite the fractions with a common denominator, and then perform addition or subtraction on the numerators. Where possible, we may also perform some cancellation to the answer.

EXAMPLE 1.4.1. Following the rules precisely, we have

$$\frac{1}{3}+\frac{1}{6} = \frac{6}{18}+\frac{3}{18} = \frac{6+3}{18} = \frac{9}{18} = \frac{1}{2}.$$

However, we can somewhat simplify the argument by using the lowest common denominator instead of the product of the denominators, and obtain

$$\frac{1}{3}+\frac{1}{6} = \frac{2}{6}+\frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}.$$

The next few examples may involve ideas discussed in the previous sections. The reader is advised to try to identify the use of the various laws discussed earlier.

EXAMPLE 1.4.2. Consider the expression

$$\frac{(x-4)^2}{(x+4)^2} - \frac{(x+2)^2}{(x+4)^2}.$$

Here the denominators are the same, so we need only perform subtraction on the numerators. We have

$$\begin{aligned}\frac{(x-4)^2}{(x+4)^2} - \frac{(x+2)^2}{(x+4)^2} &= \frac{(x-4)^2-(x+2)^2}{(x+4)^2} = \frac{(x^2-8x+16)-(x^2+4x+4)}{(x+4)^2} \\ &= \frac{x^2-8x+16-x^2-4x-4}{(x+4)^2} = \frac{12-12x}{(x+4)^2}.\end{aligned}$$

EXAMPLE 1.4.3. We have

$$\begin{aligned}\frac{3(x-1)}{x+1} + \frac{2(x+1)}{x-1} &= \frac{3(x-1)^2}{(x+1)(x-1)} + \frac{2(x+1)^2}{(x+1)(x-1)} = \frac{3(x-1)^2+2(x+1)^2}{(x+1)(x-1)} \\ &= \frac{3(x^2-2x+1)+2(x^2+2x+1)}{x^2-1} = \frac{(3x^2-6x+3)+(2x^2+4x+2)}{x^2-1} \\ &= \frac{3x^2-6x+3+2x^2+4x+2}{x^2-1} = \frac{5x^2-2x+5}{x^2-1}.\end{aligned}$$

EXAMPLE 1.4.4. We have

$$\frac{x}{y} - \frac{x}{x+y} = \frac{x(x+y)}{y(x+y)} - \frac{yx}{y(x+y)} = \frac{x(x+y)-yx}{y(x+y)} = \frac{x^2+xy-yx}{y(x+y)} = \frac{x^2+xy-xy}{y(x+y)} = \frac{x^2}{y(x+y)}.$$

EXAMPLE 1.4.5. We have

$$\frac{p}{p-q} + \frac{q}{q-p} = \frac{p}{p-q} + \frac{-q}{p-q} = \frac{p+(-q)}{p-q} = \frac{p-q}{p-q} = 1.$$

Note here that the two denominators are essentially the same, apart from a sign change. Changing the sign of both the numerator and denominator of one of the fractions has the effect of giving two fractions with the same denominator.

EXAMPLE 1.4.6. We have

$$\frac{4}{a} - \frac{2}{a(a+2)} = \frac{4(a+2)}{a(a+2)} - \frac{2}{a(a+2)} = \frac{4(a+2)-2}{a(a+2)} = \frac{(4a+8)-2}{a(a+2)} = \frac{4a+8-2}{a(a+2)} = \frac{4a+6}{a(a+2)}.$$

Note here that the common denominator is not the product of the two denominators, since we have observed the common factor a in the two denominators. If we do not make this observation, then we have

$$\begin{aligned}\frac{4}{a} - \frac{2}{a(a+2)} &= \frac{4a(a+2)}{a^2(a+2)} - \frac{2a}{a^2(a+2)} = \frac{4a(a+2)-2a}{a^2(a+2)} = \frac{(4a^2+8a)-2a}{a^2(a+2)} \\ &= \frac{4a^2+8a-2a}{a^2(a+2)} = \frac{4a^2+6a}{a^2(a+2)} = \frac{a(4a+6)}{a^2(a+2)} = \frac{4a+6}{a(a+2)}.\end{aligned}$$

Note that the common factor a is cancelled from the numerator and denominator in the last step. We still have the same answer, but a little extra work is required.

Suppose next that we wish to multiply two fractions and consider

$$\frac{a}{b} \times \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$.

MULTIPLICATION OF FRACTIONS. *We have*

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

We simply multiply the numerators and denominators separately. Where possible, we may also perform some cancellation to the answer.

EXAMPLE 1.4.7. We have

$$\frac{h}{x^2}\left(1 - \frac{h}{x+h}\right) = \frac{h}{x^2}\left(\frac{x+h}{x+h} - \frac{h}{x+h}\right) = \frac{h}{x^2} \times \frac{x+h-h}{x+h} = \frac{h}{x^2} \times \frac{x}{x+h} = \frac{hx}{x^2(x+h)} = \frac{h}{x(x+h)}.$$

EXAMPLE 1.4.8. We have

$$\left(\frac{1}{x} + \frac{1}{y}\right)(x+y) = \left(\frac{y}{xy} + \frac{x}{xy}\right)(x+y) = \frac{y+x}{xy} \times (x+y) = \frac{x+y}{xy} \times \frac{x+y}{1} = \frac{(x+y)^2}{xy}.$$

EXAMPLE 1.4.9. We have

$$\left(\frac{1}{x} - \frac{1}{y}\right)\frac{1}{x-y} = \left(\frac{y}{xy} - \frac{x}{xy}\right)\frac{1}{x-y} = \frac{y-x}{xy} \times \frac{1}{x-y} = \frac{y-x}{xy(x-y)} = \frac{-(x-y)}{xy(x-y)} = -\frac{1}{xy}.$$

EXAMPLE 1.4.10. We have

$$\frac{b-c}{bc} \times \frac{b^2}{b^2-bc} = \frac{b^2(b-c)}{bc(b^2-bc)} = \frac{b^2(b-c)}{b^2c(b-c)} = \frac{1}{c}.$$

EXAMPLE 1.4.11. We have

$$\frac{a^2}{a^2-1} \times \frac{a+1}{a} = \frac{a^2(a+1)}{a(a^2-1)} = \frac{a^2(a+1)}{a(a-1)(a+1)} = \frac{a}{a-1}.$$

EXAMPLE 1.4.12. We have

$$\frac{x+y}{x^2-4y^2} \times \frac{6y-3x}{2x+2y} = \frac{(x+y)(6y-3x)}{(x^2-4y^2)(2x+2y)} = \frac{3(x+y)(2y-x)}{2(x-2y)(x+2y)(x+y)} = -\frac{3}{2(x+2y)}.$$

Suppose finally that we wish to divide one fraction by another and consider

$$\frac{a}{b} \div \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$, $c \neq 0$ and $d \neq 0$.

DIVISION OF FRACTIONS. *We have*

$$\frac{(a/b)}{(c/d)} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

In other words, we invert the divisor and then perform multiplication instead. Where possible, we may also perform some cancellation to the answer.

REMARK. Note the special cases that

$$\frac{(a/b)}{c} = \frac{a}{bc} \qquad \text{and} \qquad \frac{a}{(c/d)} = \frac{ad}{c}.$$

EXAMPLE 1.4.13. We have

$$\begin{aligned}\left(1+\frac{1}{1+x}\right) \div \frac{4}{5(1+x)} &= \left(\frac{1+x}{1+x}+\frac{1}{1+x}\right) \div \frac{4}{5(1+x)} = \frac{1+x+1}{1+x} \div \frac{4}{5(1+x)} \\ &= \frac{2+x}{1+x} \div \frac{4}{5(1+x)} = \frac{2+x}{1+x} \times \frac{5(1+x)}{4} \\ &= \frac{5(2+x)(1+x)}{4(1+x)} = \frac{5(2+x)}{4}.\end{aligned}$$

EXAMPLE 1.4.14. We have

$$\frac{6a}{a-5} \div \frac{a+5}{a^2-25} = \frac{6a}{a-5} \div \frac{a+5}{(a+5)(a-5)} = \frac{6a}{a-5} \div \frac{1}{a-5} = \frac{6a}{a-5} \times \frac{a-5}{1} = \frac{6a(a-5)}{a-5} = 6a.$$

Alternatively, we have

$$\frac{6a}{a-5} \div \frac{a+5}{a^2-25} = \frac{6a}{a-5} \times \frac{a^2-25}{a+5} = \frac{6a(a^2-25)}{(a-5)(a+5)} = \frac{6a(a^2-25)}{a^2-25} = 6a.$$

EXAMPLE 1.4.15. We have

$$\frac{\frac{1}{x}-\frac{1}{y}}{x-y} = \left(\frac{1}{x}-\frac{1}{y}\right) \div (x-y) = \left(\frac{y}{xy}-\frac{x}{xy}\right) \times \frac{1}{x-y} = \frac{y-x}{xy} \times \frac{1}{x-y} = \frac{y-x}{xy(x-y)} = -\frac{1}{xy}.$$

EXAMPLE 1.4.16. We have

$$\begin{aligned}\frac{\frac{2}{x}+\frac{3}{y}}{\frac{1}{x}-\frac{2}{y}} &= \left(\frac{2}{x}+\frac{3}{y}\right) \div \left(\frac{1}{x}-\frac{2}{y}\right) = \left(\frac{2y}{xy}+\frac{3x}{xy}\right) \div \left(\frac{y}{xy}-\frac{2x}{xy}\right) = \frac{2y+3x}{xy} \div \frac{y-2x}{xy} \\ &= \frac{2y+3x}{xy} \times \frac{xy}{y-2x} = \frac{(2y+3x)xy}{xy(y-2x)} = \frac{2y+3x}{y-2x}.\end{aligned}$$

EXAMPLE 1.4.17. We have

$$\begin{aligned}\frac{x^2-y^2}{\frac{1}{x}+\frac{1}{y}} &= (x^2-y^2) \div \left(\frac{1}{x}+\frac{1}{y}\right) = (x^2-y^2) \div \left(\frac{y}{xy}+\frac{x}{xy}\right) = (x^2-y^2) \div \frac{y+x}{xy} \\ &= (x^2-y^2) \times \frac{xy}{x+y} = \frac{xy(x^2-y^2)}{x+y} = \frac{xy(x-y)(x+y)}{x+y} = xy(x-y).\end{aligned}$$

1.5. Factorization

Very often, we have to handle mathematical expressions that can be simplified. We have seen a few instances of cancellation of common terms in the numerator and denominator of fractions. We now consider the question of factorization. This can be thought of as the reverse process of expansion. It is difficult, if not impossible, to write down rules for factorization. Instead, we shall look at a few examples, and illustrate some of the ideas.

EXAMPLE 1.5.1. Consider the expression $x^3 - x$. First of all, we recognize that both terms x^3 and x have a factor x. Hence we can write $x^3 - x = x(x^2 - 1)$, using one of the Distributive laws. Next, we realize that we can apply one of the Laws on squares on the factor $x^2 - 1$. Hence

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1).$$

EXAMPLE 1.5.2. Consider the expression $a^4 - b^4$. Note that we can apply one of the Laws on squares to obtain $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$. We can again apply one of the Laws on squares on the factor $a^2 - b^2$. Hence

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2).$$

EXAMPLE 1.5.3. Consider the expression $x^3 - 64$. Note that $64 = 4^3$. Applying one of the Laws on cubes, we obtain $x^3 - 64 = (x - 4)(x^2 + 4x + 16)$.

Sometimes, we can factorize a quadratic polynomial $ax^2 + bx + c$. To do this, we must first study the problem of solving quadratic equations, a problem we shall consider in Section 5.2. We shall consider the problem of factorization further in Section 5.3. Here we confine ourselves to a few more examples.

EXAMPLE 1.5.4. Consider the expression $m^2 - n^2 + 4m + 4n$. We may write

$$m^2 - n^2 + 4m + 4n = m^2 + 4m + 4n - n^2 = m(m + 4) + n(4 - n),$$

and this does not lead anywhere. However, we may recognize that

$$m^2 - n^2 + 4m + 4n = (m - n)(m + n) + 4(m + n) = (m - n + 4)(m + n),$$

and this gives a good factorization.

EXAMPLE 1.5.5. We have

$$\begin{aligned} x^3 - 2x^2 - 4x + 8 &= (x^3 - 2x^2) - (4x - 8) = x^2(x - 2) - 4(x - 2) \\ &= (x^2 - 4)(x - 2) = (x - 2)(x + 2)(x - 2) = (x - 2)^2(x + 2). \end{aligned}$$

EXAMPLE 1.5.6. We have

$$\begin{aligned} \frac{a+1}{a^2 - a} - \frac{a-1}{a^2 + a} &= \frac{a+1}{a(a-1)} - \frac{a-1}{a(a+1)} = \frac{(a+1)^2}{a(a-1)(a+1)} - \frac{(a-1)^2}{a(a-1)(a+1)} = \frac{(a+1)^2 - (a-1)^2}{a(a-1)(a+1)} \\ &= \frac{((a+1) - (a-1))((a+1) + (a-1))}{a(a-1)(a+1)} = \frac{(a + 1 - a + 1)(a + 1 + a - 1)}{a(a-1)(a+1)} \\ &= \frac{4a}{a(a-1)(a+1)} = \frac{4}{(a-1)(a+1)}. \end{aligned}$$

EXAMPLE 1.5.7. We have

$$\begin{aligned} \frac{2}{x^2 - 1} - \frac{1}{x^2 - x} + \frac{x-1}{x^2 + x} &= \frac{2}{(x-1)(x+1)} - \frac{1}{x(x-1)} + \frac{x-1}{x(x+1)} \\ &= \frac{2x}{x(x-1)(x+1)} - \frac{x+1}{x(x-1)(x+1)} + \frac{(x-1)^2}{x(x-1)(x+1)} \\ &= \frac{2x - (x+1) + (x-1)^2}{x(x-1)(x+1)} = \frac{2x - x - 1 + x^2 - 2x + 1}{x(x-1)(x+1)} \\ &= \frac{x^2 - x}{x(x-1)(x+1)} = \frac{x(x-1)}{x(x-1)(x+1)} = \frac{1}{x+1}. \end{aligned}$$

Problems for Chapter 1

1. Find the precise value of each of the following expressions:
   a) $5 + 4 \times 3 \div 2 - 1$
   b) $(1 + 2) \times 3 - 4 \div 5$
   c) $(54 \div 3 + 18 \times 2) \div 9$
   d) $4 + 2 - 4 + 5 \times (-2) \times (1 + 3)$
   e) $2 + 5 \times (-1) - (2 + 3) \times 4 \div 10 + 4 - (3 - 5)$
   f) $((4 + 2) \times 3 + 1) \times 5 + 10 \div 2$
   g) $\sqrt{(-4) \times (2 - 11)}$
   h) $-\sqrt{5^2 + 12^2}$
   i) $\sqrt{5 \times 5 - 4 \times 4} - \sqrt{3 \times 3 - 2 \times 2 - (-1) \times (-1)}$

2. Expand each of the following expressions:
   a) $(4x + 3)(5x - 2)$
   b) $(4x + 3)^2 + (5x - 2)^2$
   c) $(4x + 3)^2 - (5x - 2)^2$
   d) $(7x - 2)^2 + (4x + 5)^2$
   e) $(x + y)(x - 2y)$
   f) $(x + 2y + 3)(2x - y - 1)$
   g) $(x + 2y)(x - 2y)^2$
   h) $(x + 2y)^2(x - 2y)^2$

3. Rewrite each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{3}{4} + \dfrac{2}{3}$
   b) $\dfrac{5}{6} - \dfrac{1}{12}$
   c) $\dfrac{5x}{x + 2} + \dfrac{3x}{x + 4}$
   d) $\dfrac{3}{x - 1} - \dfrac{3}{x + 1}$
   e) $\dfrac{5}{x} + \dfrac{3}{x(x + 1)}$
   f) $\dfrac{(x + y)^2}{x^2} - \dfrac{(x - y)^2}{y^2}$

4. Rewrite each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{2 + 3}{4 + 5} \times \dfrac{6 + 7}{8 + 9}$
   b) $\dfrac{2 + 3}{4 + 5} \div \dfrac{6 + 7}{8 + 9}$
   c) $\left(\dfrac{1}{2} + \dfrac{1}{3}\right) \div \left(\dfrac{3}{4} + \dfrac{4}{3}\right) \times \left(\dfrac{5}{14} + \dfrac{3}{2} \times \dfrac{3}{7} + \dfrac{3}{2}\right)$
   d) $\dfrac{x}{y^2} \times \dfrac{xy - yz}{x}$
   e) $\dfrac{x^2 - y^2}{x + y} \div \dfrac{x - y}{x^3 + y^3}$
   f) $\left(\dfrac{4}{x} - \dfrac{3}{y}\right) \div \left(\dfrac{5}{x} + \dfrac{6}{y}\right)$

5. Factorize each of the following expressions, using the laws on squares and cubes as necessary:
   a) $x^4 - x^2$
   b) $x^6 - y^6$
   c) $x^3 y - x y^3$
   d) $x^5 y^2 + x^2 y^5$

6. Simplify each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{3}{x(x + 2)} + \dfrac{1}{x^2 - 2x} - \dfrac{2}{x^2 - 4}$
   b) $\dfrac{1}{x^2 + xy} + \dfrac{1}{y^2 + xy}$
   c) $\dfrac{x}{x - y} - \dfrac{y}{x + y} - \dfrac{2xy}{x^2 - y^2}$
   d) $\left(x + \dfrac{y^2}{x - y}\right) \div \dfrac{x^3 + y^3}{x^2 - y^2}$
   e) $x^3 - y^3 + x^2 y - x y^2$

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