ELEMENTARY MATHEMATICS

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Chapter 2

INTRODUCTION TO MATRICES

2.1. Linear Equations

EXAMPLE 2.1.1. Consider the two linear equations

$$\begin{aligned} 3x + 4y &= 11, \\ 5x + 7y &= 19. \end{aligned}$$

It is easy to check the two equations are satisfied when $x = 1$ and $y = 2$. We can represent these two linear equations in matrix form as

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix},$$

where we adopt the convention that

$$\begin{pmatrix} 3 & 4 \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ \bullet \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \bullet & \bullet \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bullet \\ 19 \end{pmatrix}$$

represent respectively the information $3x + 4y = 11$ and $5x + 7y = 19$. Under this convention, it is easy to see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for every $x, y \in \mathbb{R}$. Next, observe that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

† This chapter was written at Macquarie University in 1999.

where, under a convention slightly more general to the one used earlier, we have

$$\begin{pmatrix} 7 & -4 \\ \bullet & \bullet \end{pmatrix}\begin{pmatrix} 3 & \bullet \\ 5 & \bullet \end{pmatrix} = \begin{pmatrix} 1 & \bullet \\ \bullet & \bullet \end{pmatrix}, \qquad \begin{pmatrix} 7 & -4 \\ \bullet & \bullet \end{pmatrix}\begin{pmatrix} \bullet & 4 \\ \bullet & 7 \end{pmatrix} = \begin{pmatrix} \bullet & 0 \\ \bullet & \bullet \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & \bullet \\ 5 & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ 0 & \bullet \end{pmatrix}, \qquad \begin{pmatrix} \bullet & \bullet \\ -5 & 3 \end{pmatrix}\begin{pmatrix} \bullet & 4 \\ \bullet & 7 \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ \bullet & 1 \end{pmatrix},$$

representing respectively $(7 \times 3) + ((-4) \times 5) = 1$, $(7 \times 4) + ((-4) \times 7) = 0$, $((-5) \times 3) + (3 \times 5) = 0$ and $((-5) \times 4) + (3 \times 7) = 1$. It now follows on the one hand that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and on the other hand that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 11 \\ 19 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The convention mentioned in the example above is simply the rule concerning the multiplication of matrices. The purpose of this chapter is to study the arithmetic in connection with matrices. We shall be concerned primarily with 2×2 real matrices. These are arrays of real numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

consisting of two rows counted from top to bottom, and two columns counted from left to right. An entry a_{ij} thus corresponds to the entry in row i and column j.

2.2. Arithmetic

ADDITION AND SUBTRACTION. *Suppose that*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are two 2×2 matrices. Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \qquad \text{and} \qquad A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}.$$

In other words, we perform addition and subtraction entrywise. The operations addition and subtraction are governed by the following rules:

(a) Operations within brackets are performed first.
(b) Addition and subtraction are performed in their order of appearance.
(c) A number of additions can be performed in any order. For any 2×2 matrices A, B, C, we have $A + (B + C) = (A + B) + C$ and $A + B = B + A$.

Example 2.2.1. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix},$$

and

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix}.$$

Example 2.2.2. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \right) + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 9 & 14 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix}.$$

Example 2.2.3. Like real numbers, it is not true in general that $A - (B - C) = (A - B) - C$. Note that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right) = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \right) - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}.$$

Remark. The matrix

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies $0 + A = A + 0 = A$ for any 2×2 matrix A, and plays a role analogous to the real number 0 in addition of real numbers. This matrix 0 is called the zero matrix.

MULTIPLICATION BY A REAL NUMBER. *Suppose that*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a 2×2 matrix, and that r is a real number. Then

$$rA = \begin{pmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{pmatrix}.$$

In other words, we multiply each entry of A by the same real number r. This operation is governed by the following rules:
- *(a) Operations within brackets are performed first.*
- *(b) If there are no brackets to indicate priority, then multiplication by a real number takes precedence over addition and subtraction.*
- *(c) A number of multiplications by real numbers can be performed in any order. For any 2×2 matrix A and any real numbers $r, s \in \mathbb{R}$, we have $(rs)A = r(sA)$.*
- *(d) For any 2×2 matrix A and any real numbers $r, s \in \mathbb{R}$, we have $(r + s)A = rA + sA$.*
- *(e) For any 2×2 matrices A, B and any real number $r \in \mathbb{R}$, we have $r(A + B) = rA + rB$.*

Example 2.2.4. We have

$$2 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = 2 \begin{pmatrix} 4 & 6 \\ 8 & 11 \end{pmatrix} = \begin{pmatrix} 8 & 12 \\ 16 & 22 \end{pmatrix}.$$

Example 2.2.5. We have

$$3 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = 3 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \right) = 3 \begin{pmatrix} 5 & 8 \\ 11 & 15 \end{pmatrix} = \begin{pmatrix} 15 & 24 \\ 33 & 45 \end{pmatrix}.$$

Example 2.2.6. We have

$$2\left(3\begin{pmatrix}3&4\\5&7\end{pmatrix}\right)=2\begin{pmatrix}9&12\\15&21\end{pmatrix}=\begin{pmatrix}18&24\\30&42\end{pmatrix}=6\begin{pmatrix}3&4\\5&7\end{pmatrix}=(2\times 3)\begin{pmatrix}3&4\\5&7\end{pmatrix}.$$

Example 2.2.7. We have

$$5\begin{pmatrix}3&4\\5&7\end{pmatrix}-2\begin{pmatrix}3&4\\5&7\end{pmatrix}=\begin{pmatrix}15&20\\25&35\end{pmatrix}-\begin{pmatrix}6&8\\10&14\end{pmatrix}=\begin{pmatrix}9&12\\15&21\end{pmatrix}=3\begin{pmatrix}3&4\\5&7\end{pmatrix}=(5-2)\begin{pmatrix}3&4\\5&7\end{pmatrix}.$$

Example 2.2.8. We have

$$3\left(\begin{pmatrix}3&4\\5&7\end{pmatrix}-\begin{pmatrix}1&2\\3&4\end{pmatrix}\right)=3\begin{pmatrix}2&2\\2&3\end{pmatrix}=\begin{pmatrix}6&6\\6&9\end{pmatrix},$$

and

$$3\begin{pmatrix}3&4\\5&7\end{pmatrix}-3\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}9&12\\15&21\end{pmatrix}-\begin{pmatrix}3&6\\9&12\end{pmatrix}=\begin{pmatrix}6&6\\6&9\end{pmatrix}.$$

MULTIPLICATION OF MATRICES. *Suppose that*

$$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}$$

are 2×2 *matrices. Then*

$$AB=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}&a_{11}b_{12}+a_{12}b_{22}\\a_{21}b_{11}+a_{22}b_{21}&a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}.$$

This operation is governed by the following rules:
(a) Operations within brackets are performed first.
(b) If there are no brackets to indicate priority, then multiplication takes precedence over addition and subtraction.
(c) For any 2×2 *matrices* A,B,C*, we have* $(AB)C=A(BC)$.
(d) For any 2×2 *matrices* A,B *and any real numbers* $r\in\mathbb{R}$*, we have* $r(AB)=(rA)B=A(rB)$.
(e) For any 2×2 *matrices* A,B,C*, we have* $A(B+C)=AB+AC$ *and* $(A+B)C=AC+BC$.

Remarks. (1) Note that the definition above agrees with the convention adopted in Example 2.1.1. Observe that we have

$$\begin{pmatrix}a_{11}&a_{12}\\\bullet&\bullet\end{pmatrix}\begin{pmatrix}b_{11}&\bullet\\b_{21}&\bullet\end{pmatrix}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}&\bullet\\\bullet&\bullet\end{pmatrix},$$

$$\begin{pmatrix}a_{11}&a_{12}\\\bullet&\bullet\end{pmatrix}\begin{pmatrix}\bullet&b_{12}\\\bullet&b_{22}\end{pmatrix}=\begin{pmatrix}\bullet&a_{11}b_{12}+a_{12}b_{22}\\\bullet&\bullet\end{pmatrix},$$

$$\begin{pmatrix}\bullet&\bullet\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}&\bullet\\b_{21}&\bullet\end{pmatrix}=\begin{pmatrix}\bullet&\bullet\\a_{21}b_{11}+a_{22}b_{21}&\bullet\end{pmatrix},$$

$$\begin{pmatrix}\bullet&\bullet\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\bullet&b_{12}\\\bullet&b_{22}\end{pmatrix}=\begin{pmatrix}\bullet&\bullet\\\bullet&a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}.$$

(2) The matrix

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

satisfies $IA=AI=A$ for any 2×2 matrix A, and plays a role analogous to the real number 1 in multiplication of real numbers. This matrix I is called the identity matrix.

(3) Multiplication of matrices is generally not commutative; in other words, given two 2×2 matrices A and B, it is not automatic that $AB = BA$. For example, let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix}.$$

Example 2.2.9. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right)\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -2 & 19 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\left(\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -12 & 19 \end{pmatrix}.$$

Example 2.2.10. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}+\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right)\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}+\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ -10 & 9 \end{pmatrix}+\begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix}.$$

Since I is the identity matrix, we would like to find a technique to obtain, for any given 2×2 matrix A, an inverse 2×2 matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. This is not always possible, since in the case of real numbers, the number 0 does not have a multiplicative inverse. We therefore need a condition on 2×2 matrices which is equivalent to saying that a real number is non-zero.

MULTIPLICATIVE INVERSE. *For any 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying the condition $ad - bc \neq 0$, the matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

satisfies $AA^{-1} = A^{-1}A = I$. In this case, we say that A is invertible with multiplicative inverse A^{-1}.

Remarks. (1) The quantity $ad - bc$ is known as the determinant of the matrix A. The result above says that any 2×2 matrix is invertible as long as it has non-zero determinant.

(2) If two 2×2 matrices A and B both have non-zero determinants, then it can be shown that the matrix product AB also has non-zero determinant. We also have $(AB)^{-1} = B^{-1}A^{-1}$.

Example 2.2.11. Recall Example 2.1.1. It is easy to check that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

EXAMPLE 2.2.12. The matrix $\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ has determinant 0 and so is not invertible.

EXAMPLE 2.2.13. Consider the matrices

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \qquad \text{and} \qquad B^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

Note also that

$$AB = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 18 & 33 \\ 31 & 57 \end{pmatrix}.$$

We have

$$(AB)^{-1} = \frac{1}{3}\begin{pmatrix} 57 & -33 \\ -31 & 18 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = B^{-1}A^{-1}.$$

2.3. Application to Linear Equations

We now return to the problem first discussed in Section 2.1. Consider the two linear equations

$$\begin{aligned} ax + by &= s, \\ cx + dy &= t, \end{aligned}$$

where $a, b, c, d, s, t \in \mathbb{R}$ are given and x and y are the unknowns.

We can represent these two linear equations in matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}.$$

If $ad - bc \neq 0$, then the 2×2 matrix on the left hand side is invertible. It follows that there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix},$$

giving the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}.$$

EXAMPLE 2.3.1. Suppose that

$$\begin{aligned} x + y &= 32, \\ 3x + 2y &= 70. \end{aligned}$$

The two linear equations can be represented in matrix form

$$\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ 70 \end{pmatrix}.$$

Note now that the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ has determinant -1 and multiplicative inverse $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$. It follows that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 32 \\ 70 \end{pmatrix} = \begin{pmatrix} 6 \\ 26 \end{pmatrix},$$

giving the solution $x = 6$ and $y = 26$.

We shall discuss a different technique for solving such equations in Section 5.1.

Problems for Chapter 2

1. Write each of the following systems of linear equations in matrix form:
   a) $3x - 8y = 1$, $2x + 3y = 9$
   b) $4x - 3y = 14$, $9x - 4y = 26$
   c) $6x - 2y = 14$, $2x + 3y = 12$
   d) $5x + 2y = 4$, $7x + 3y = 5$

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 3 \\ 5 & 1 \end{pmatrix}.$$

   a) Verify that $(A + B) + C = A + (B + C)$ and $A + B = B + A$.
   b) Find $A + 3B$ and $7A - 2B + 3C$.
   c) Verify that $(AB)C = A(BC)$.
   d) Is it true that $AB = BA$? Comment on the result.
   e) Find A^{-1} and B^{-1}.
   f) Find $(AB)^{-1}$, and verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Solve each of the systems of linear equations in Question 1 by using inverse matrices.

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