

ELEMENTARY MATHEMATICS

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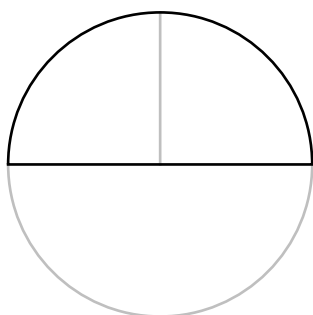
Chapter 3

TRIGONOMETRY

3.1. Radian and Arc Length

The number π plays a central role in the study of trigonometry. We all know that a circle of radius 1 has area π and circumference 2π . It is also very useful in describing angles, as we shall now show.

Let us split a circle of radius 1 along a diameter into two semicircles as shown in the picture below.



The circumference of the circle is now split into two equal parts, each of length π and each subtending an angle 180° . If we use the convention that $\pi = 180^\circ$, then the arc of the semicircle of radius 1 will be the same as the angle it subtends. If we further split the arc of the semicircle of radius 1 into two equal parts, then each of the two parts forms an arc of length $\pi/2$ and subtends an angle $90^\circ = \pi/2$. In fact, any arc of a circle of radius 1 which subtends an angle θ must have length θ under our convention.

We now formalize our discussion so far.

DEFINITION. An angle of 1 radian is defined to be the angle subtended by an arc of length 1 on a circle of radius 1.

† This chapter was written at Macquarie University in 1999.

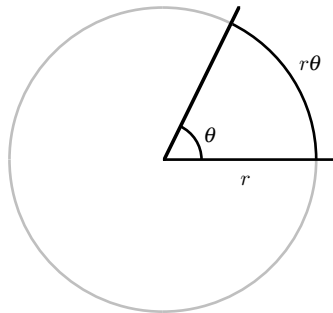
REMARKS. (1) Very often, the term radian is omitted when we discuss angles. We simply refer to an angle 1 or an angle π , rather than an angle of 1 radian or an angle of π radian.

(2) Simple calculation shows that 1 radian is equal to $(180/\pi)^\circ = 57.2957795\dots^\circ$. Similarly, we can show that 1° is equal to $(\pi/180)$ radian = 0.01745329... radian. In fact, since π is irrational, the digits do not terminate or repeat.

(3) We observe the following special values:

$$\frac{\pi}{6} = 30^\circ, \quad \frac{\pi}{4} = 45^\circ, \quad \frac{\pi}{3} = 60^\circ, \quad \frac{\pi}{2} = 90^\circ, \quad \pi = 180^\circ, \quad 2\pi = 360^\circ.$$

Consider now a circle of radius r and an angle θ given in radian, as shown in the picture below.



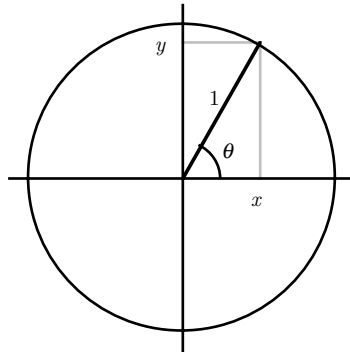
Clearly the length s of the arc which subtends the angle θ satisfies $s = r\theta$, while the area A of the sector satisfies

$$A = \pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

Note that πr^2 is equal to the area inside the circle, while $\theta/2\pi$ is the proportion of the area in question.

3.2. The Trigonometric Functions

Consider the xy -plane, together with a circle of radius 1 and centred at the origin $(0,0)$. Suppose that θ is an angle measured anticlockwise from the positive x -axis, and the point (x,y) on the circle is as shown in the picture below.



We define

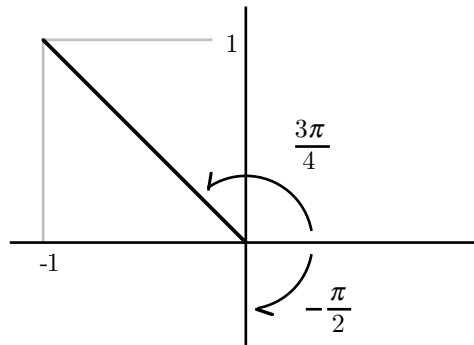
$$\cos \theta = x \quad \text{and} \quad \sin \theta = y.$$

Furthermore, we define

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{x}{y}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{x} \quad \text{and} \quad \csc \theta = \frac{1}{\sin \theta} = \frac{1}{y}.$$

REMARKS. (1) Note that $\tan \theta$ and $\sec \theta$ are defined only when $\cos \theta \neq 0$, and that $\cot \theta$ and $\csc \theta$ are defined only when $\sin \theta \neq 0$.

(2) It is a good habit to always measure an angle from the positive x -axis, using the convention that positive angles are measured anticlockwise and negative angles are measured clockwise, as illustrated below.



(3) We then observe that

$$\cos \theta = \frac{\text{horizontal side}}{\text{hypotenuse}} \quad \text{and} \quad \sin \theta = \frac{\text{vertical side}}{\text{hypotenuse}},$$

as well as

$$\tan \theta = \frac{\text{vertical side}}{\text{horizontal side}} \quad \text{and} \quad \cot \theta = \frac{\text{horizontal side}}{\text{vertical side}},$$

with the convention that

$$\text{horizontal side} \begin{cases} > 0 & \text{(to the right of the (vertical) } y\text{-axis),} \\ < 0 & \text{(to the left of the (vertical) } y\text{-axis),} \end{cases}$$

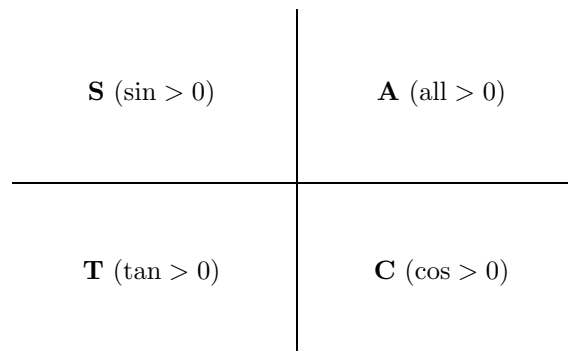
and

$$\text{vertical side} \begin{cases} > 0 & \text{(above the (horizontal) } x\text{-axis),} \\ < 0 & \text{(below the (horizontal) } x\text{-axis),} \end{cases}$$

while

$$\text{hypotenuse} > 0 \quad (\text{always}).$$

(4) It is also useful to remember the CAST rule concerning sine, cosine and tangent.



PYTHAGOREAN IDENTITIES. For every value $\theta \in \mathbb{R}$ for which the trigonometric functions in question are defined, we have

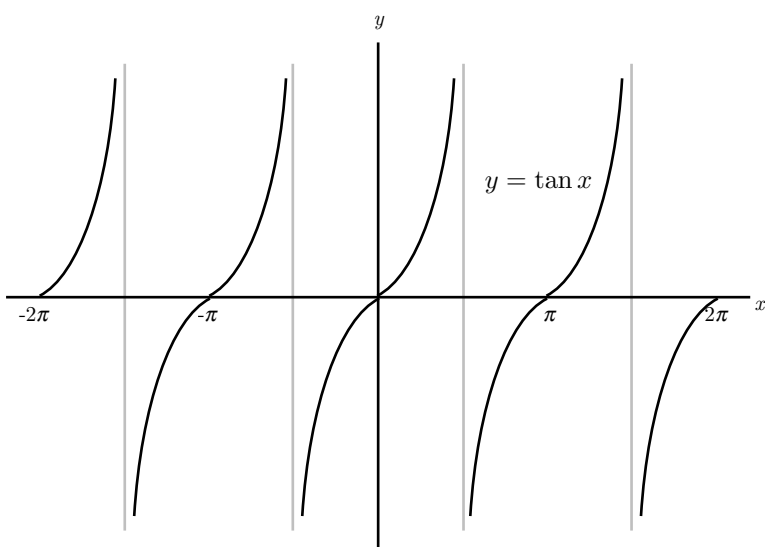
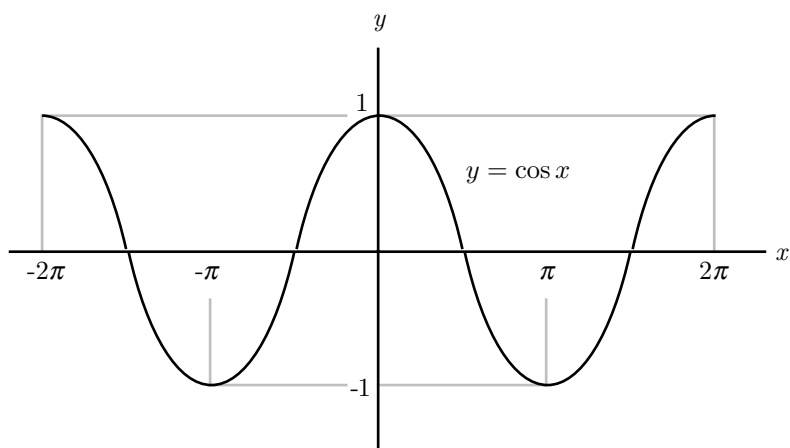
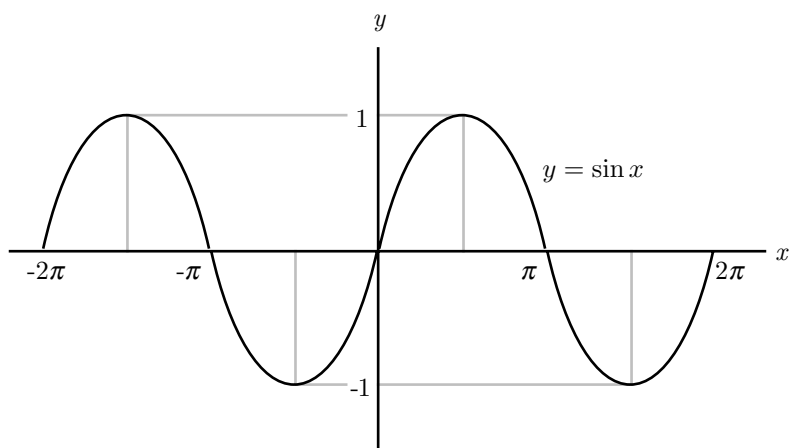
(a) $\cos^2 \theta + \sin^2 \theta = 1$;

(b) $1 + \tan^2 \theta = \sec^2 \theta$; and

(c) $1 + \cot^2 \theta = \csc^2 \theta$.

PROOF. Note that (a) follows from the classical Pythagoras's theorem. If $\cos \theta \neq 0$, then dividing both sides of (a) by $\cos^2 \theta$ gives (b). If $\sin \theta \neq 0$, then dividing both sides of (a) by $\sin^2 \theta$ gives (c). ♣

We sketch the graphs of the functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $-2\pi \leq x \leq 2\pi$ below:



We shall use these graphs to make some observations about trigonometric functions.

PROPERTIES OF TRIGONOMETRIC FUNCTIONS.

(a) The functions $\sin x$ and $\cos x$ are periodic with period 2π . More precisely, for every $x \in \mathbb{R}$, we have

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x.$$

(b) The functions $\tan x$ and $\cot x$ are periodic with period π . More precisely, for every $x \in \mathbb{R}$ for which the trigonometric function in question is defined, we have

$$\tan(x + \pi) = \tan x \quad \text{and} \quad \cot(x + \pi) = \cot x.$$

(c) The function $\sin x$ is an odd function, and the function $\cos x$ is an even function. More precisely, for every $x \in \mathbb{R}$, we have

$$\sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x.$$

(d) The functions $\tan x$ and $\cot x$ are odd functions. More precisely, for every $x \in \mathbb{R}$ for which the trigonometric function in question is defined, we have

$$\tan(-x) = -\tan x \quad \text{and} \quad \cot(-x) = -\cot x.$$

(e) For every $x \in \mathbb{R}$, we have

$$\sin(x + \pi) = -\sin x \quad \text{and} \quad \cos(x + \pi) = -\cos x.$$

(f) For every $x \in \mathbb{R}$, we have

$$\sin(\pi - x) = \sin x \quad \text{and} \quad \cos(\pi - x) = -\cos x.$$

(g) For every $x \in \mathbb{R}$ for which the trigonometric function in question is defined, we have

$$\tan(\pi - x) = -\tan x \quad \text{and} \quad \cot(\pi - x) = -\cot x.$$

(h) For every $x \in \mathbb{R}$, we have

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x \quad \text{and} \quad \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

(i) For every $x \in \mathbb{R}$, we have

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \text{and} \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

(j) For every $x \in \mathbb{R}$ for which the trigonometric functions in question are defined, we have

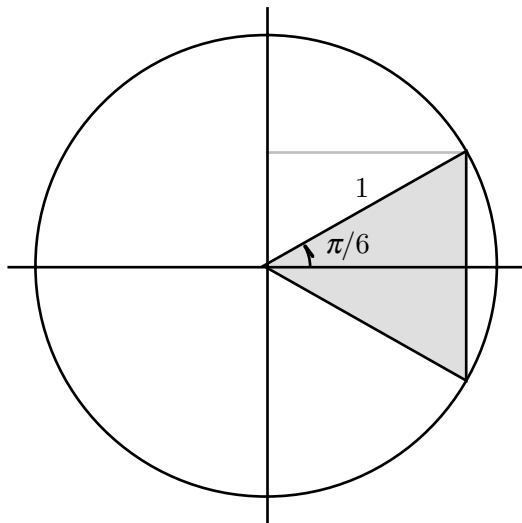
$$\tan\left(x + \frac{\pi}{2}\right) = -\cot x \quad \text{and} \quad \cot\left(x + \frac{\pi}{2}\right) = -\tan x.$$

(k) For every $x \in \mathbb{R}$ for which the trigonometric function in question is defined, we have

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \quad \text{and} \quad \cot\left(\frac{\pi}{2} - x\right) = \tan x.$$

REMARK. There is absolutely no need to remember any of these properties! We shall discuss later some trigonometric identities which will give all the above as special cases.

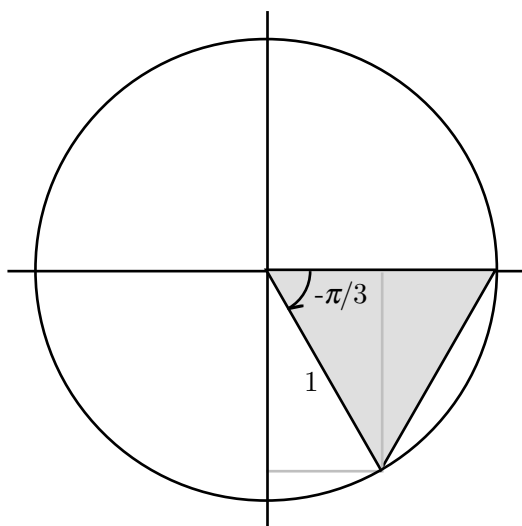
EXAMPLE 3.2.1. Consider the following picture.



Clearly the triangle shown is an equilateral triangle, with all three sides of equal length. It is also clear that $\sin(\pi/6)$ is half the length of the vertical side. It follows that we must have $\sin(\pi/6) = 1/2$. To find the precise value for $\cos(\pi/6)$, we first observe that $\cos(\pi/6) > 0$. On the other hand, it follows from the first of the Pythagorean identities that $\cos^2(\pi/6) = 3/4$. Hence $\cos(\pi/6) = \sqrt{3}/2$. We can then deduce that $\tan(\pi/6) = 1/\sqrt{3}$.

EXAMPLE 3.2.2. We have $\tan(13\pi/6) = \tan(7\pi/6) = \tan(\pi/6) = 1/\sqrt{3}$. Note that we have used part (b) of the Properties of trigonometric functions, as well as the result from Example 3.2.1.

EXAMPLE 3.2.3. Consider the following picture.



Clearly the triangle shown is an equilateral triangle, with all three sides of equal length. It is also clear that $\cos(-\pi/3)$ is half the length the horizontal side. It follows that we must have $\cos(-\pi/3) = 1/2$. To find the precise value for $\sin(-\pi/3)$, we first observe that $\sin(-\pi/3) < 0$. On the other hand, it follows from the first of the Pythagorean identities that $\sin^2(-\pi/3) = 3/4$. Hence $\sin(-\pi/3) = -\sqrt{3}/2$. Alternatively, we can deduce from Example 3.2.1 by using parts (c) and (i) of the Properties of

trigonometric functions that

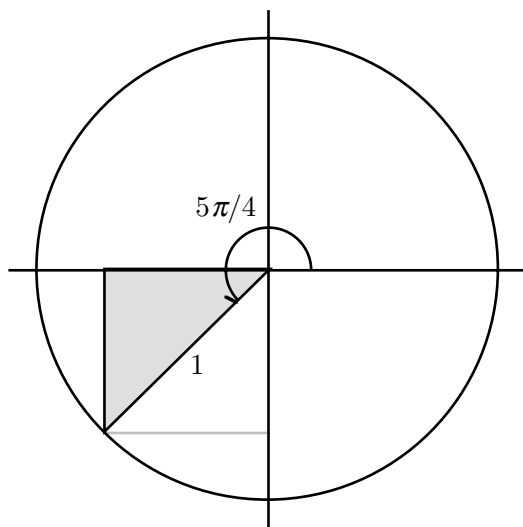
$$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

and

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$$

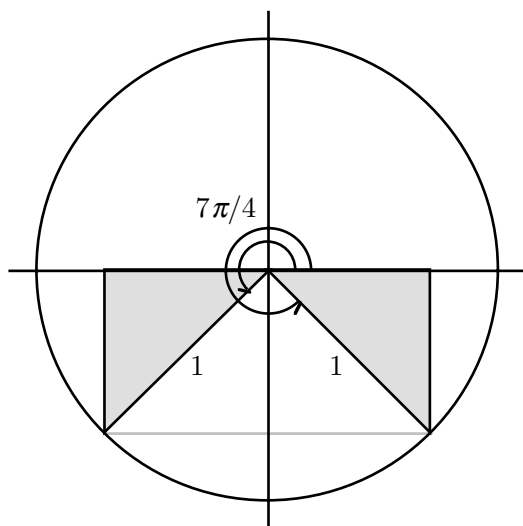
We can then deduce that $\tan(-\pi/3) = -\sqrt{3}$.

EXAMPLE 3.2.4. Consider the following picture.



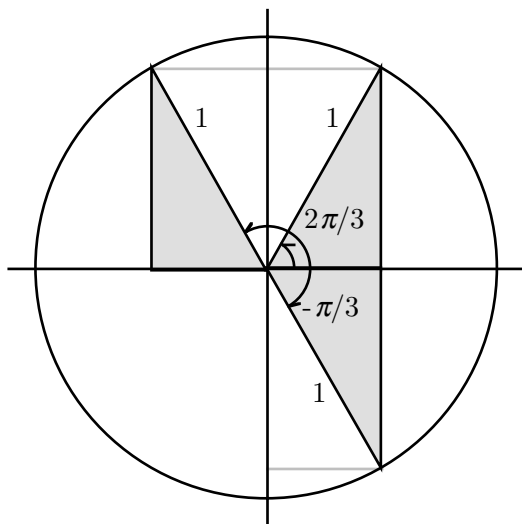
Clearly the triangle shown is a right-angled triangle with the two shorter sides of equal length and hypotenuse of length 1. It is also clear that $\sin(5\pi/4)$ is the length of the vertical side, with a $-$ sign attached as it is below the horizontal axis. On the other hand, it is also clear that $\cos(5\pi/4)$ is the length of the horizontal side, again with a $-$ sign attached as it is left of the vertical axis. It follows from Pythagoras's theorem that $\sin(5\pi/4) = \cos(5\pi/4) = y$, where $y < 0$ and $\sqrt{y^2 + y^2} = 1$. Clearly $y = -1/\sqrt{2}$. We can then deduce that $\tan(5\pi/4) = 1$.

EXAMPLE 3.2.5. To find all the solutions of the equation $\sin x = -1/\sqrt{2}$ in the interval $0 \leq x < 2\pi$, we consider the following picture.



Using Pythagoras's theorem, it is easy to see that the two triangles shown both have horizontal side of length $1/\sqrt{2}$, the same as the length of their vertical sides. Clearly $x = 5\pi/4$ or $x = 7\pi/4$.

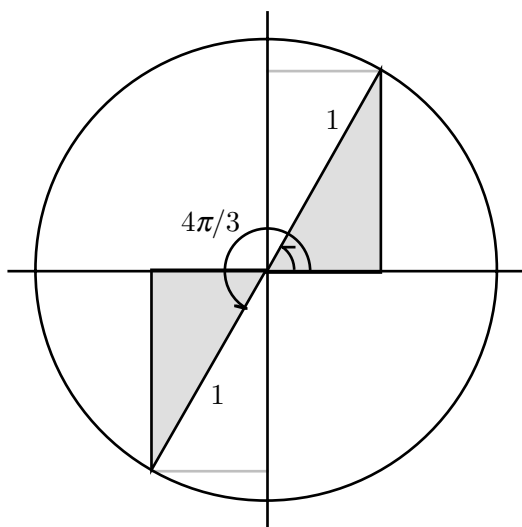
EXAMPLE 3.2.6. Suppose that we wish to find all the solutions of the equation $\cos^2 x = 1/4$ in the interval $-\pi/2 < x \leq \pi$. Observe first of all that either $\cos x = 1/2$ or $\cos x = -1/2$. We consider the following picture.



Using Pythagoras's theorem, it is easy to see that the three triangles shown both have vertical side of length $\sqrt{3}/2$. Clearly $x = -\pi/3$, $x = \pi/3$ or $x = 2\pi/3$.

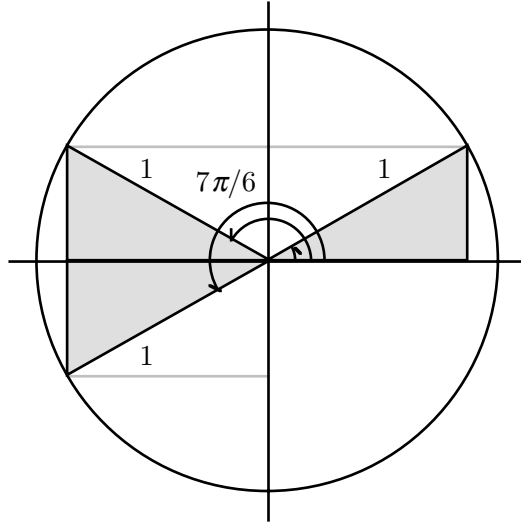
EXAMPLE 3.2.7. Convince yourself that the only two solutions of the equation $\cos^2 x = 1$ in the interval $0 \leq x < 2\pi$ are $x = 0$ and $x = \pi$.

EXAMPLE 3.2.8. To find all the solutions of the equation $\tan x = \sqrt{3}$ in the interval $0 \leq x < 2\pi$, we consider the following picture.



Since $\tan x > 0$, it follows from the CAST rule that we can restrict our attention to the first and third quadrants. It is easy to check that the two triangles shown have horizontal sides of length $1/2$ and vertical sides of length $\sqrt{3}/2$. Clearly $x = \pi/3$ or $x = 4\pi/3$.

EXAMPLE 3.2.9. Suppose that we wish to find all the solutions of the equation $\tan^2 x = 1/3$ in the interval $0 \leq x \leq 3\pi/2$. Observe first of all that either $\tan x = 1/\sqrt{3}$ or $\tan x = -1/\sqrt{3}$. We consider the following picture.



It is easy to check that the three triangles shown have horizontal sides of length $\sqrt{3}/2$ and vertical sides of length $1/2$. Clearly $x = \pi/6$, $x = 5\pi/6$ or $x = 7\pi/6$.

EXAMPLE 3.2.10. Convince yourself that the only two solutions of the equation $\sec x = \sqrt{2}$ in the interval $0 \leq x < 2\pi$ are $x = \pi/4$ and $x = 7\pi/4$.

EXAMPLE 3.2.11. For every $x \in \mathbb{R}$, we have

$$\sin^3 x + \sin x \cos^2 x = \sin x \sin^2 x + \sin x \cos^2 x = (\sin x)(\sin^2 x + \cos^2 x) = \sin x,$$

in view of the first of the Pythagorean identities.

EXAMPLE 3.2.12. For every $x \in \mathbb{R}$ such that $\cos x \neq 0$, we have

$$(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x = 1,$$

in view of the second of the Pythagorean identities.

EXAMPLE 3.2.13. For every $x \in \mathbb{R}$ such that $\cos x \neq \pm 1$, we have

$$\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} = \frac{(1 + \cos x) + (1 - \cos x)}{(1 - \cos x)(1 + \cos x)} = \frac{2}{1 - \cos^2 x} = \frac{2}{\sin^2 x} = 2 \csc^2 x.$$

EXAMPLE 3.2.14. For every $x \in \mathbb{R}$, we have

$$\begin{aligned} (\cos x + \sin x)^2 + (\cos x - \sin x)^2 &= (\cos^2 x + 2 \cos x \sin x + \sin^2 x) + (\cos^2 x - 2 \cos x \sin x + \sin^2 x) \\ &= (1 + 2 \cos x \sin x) + (1 - 2 \cos x \sin x) = 2. \end{aligned}$$

EXAMPLE 3.2.15. For every $x \in \mathbb{R}$ such that the expression on the left hand side makes sense, we have

$$\begin{aligned} \frac{1 + \cot x}{\csc x} - \frac{1 + \tan x}{\sec x} &= (1 + \cot x) \sin x - (1 + \tan x) \cos x = \left(1 + \frac{\cos x}{\sin x}\right) \sin x - \left(1 + \frac{\sin x}{\cos x}\right) \cos x \\ &= (\sin x + \cos x) - (\cos x + \sin x) = 0. \end{aligned}$$

EXAMPLE 3.2.16. Let us return to Example 3.2.5 where we showed that the solutions of the equation $\sin x = -1/\sqrt{2}$ in the interval $0 \leq x < 2\pi$ are given by $x = 5\pi/4$ and $x = 7\pi/4$. Suppose now that we wish to find all the values $x \in \mathbb{R}$ that satisfy the same equation. To do this, we can use part (a) of the Properties of trigonometric functions, and conclude that the solutions are given by

$$x = \frac{5\pi}{4} + 2k\pi \quad \text{or} \quad x = \frac{7\pi}{4} + 2k\pi,$$

where $k \in \mathbb{Z}$.

EXAMPLE 3.2.17. Consider the equation $\sec(x/2) = 2$. Then $\cos(x/2) = 1/2$. If we first restrict our attention to $0 \leq x/2 < 2\pi$, then it is not difficult to see that the solutions are given by $x/2 = \pi/3$ and $x/2 = 5\pi/3$. Using part (a) of the Properties of trigonometric functions, we conclude that without the restriction $0 \leq x/2 < 2\pi$, the solutions are given by

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \frac{x}{2} = \frac{5\pi}{3} + 2k\pi,$$

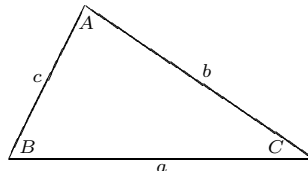
where $k \in \mathbb{Z}$. It follows that

$$x = \frac{2\pi}{3} + 4k\pi \quad \text{or} \quad x = \frac{10\pi}{3} + 4k\pi,$$

where $k \in \mathbb{Z}$.

3.3. Some Trigonometric Identities

Consider a triangle with side lengths and angles as shown in the picture below:



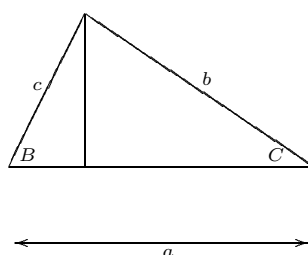
SINE RULE. We have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

COSINE RULE. We have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A, \\ b^2 &= a^2 + c^2 - 2ac \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}$$

SKETCH OF PROOF. Consider the picture below.



Clearly the length of the vertical line segment is given by $c \sin B = b \sin C$, so that

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

This gives the sine rule. Next, note that the horizontal side of the the right-angled triangle on the left has length $c \cos B$. It follows that the horizontal side of the right-angled triangle on the right has length $a - c \cos B$. If we now apply Pythagoras's theorem to this latter triangle, then we have

$$(a - c \cos B)^2 + (c \sin B)^2 = b^2,$$

so that

$$a^2 - 2ac \cos B + c^2 \cos^2 B + c^2 \sin^2 B = b^2,$$

whence

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

This gives the cosine rule. ♣

We mentioned earlier that there is no need to remember any of the Properties of trigonometric functions discussed in the last section. The reason is that they can all be deduced easily from the identities below.

SUM AND DIFFERENCE IDENTITIES. For every $A, B \in \mathbb{R}$, we have

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad \text{and} \quad \sin(A - B) = \sin A \cos B - \cos A \sin B,$$

as well as

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \text{and} \quad \cos(A - B) = \cos A \cos B + \sin A \sin B.$$

REMARKS. (1) Proofs can be sketched for these identities by drawing suitable pictures, although such pictures are fairly complicated. We omit the proofs here.

(2) It is not difficult to remember these identities. Observe the pattern

$$\sin \pm = \sin \cos \pm \cos \sin \quad \text{and} \quad \cos \pm = \cos \cos \mp \sin \sin.$$

For small positive angles, increasing the angle increases the sine (thus keeping signs) and decreases the cosine (thus reversing signs).

(3) One can also deduce analogous identities for tangent and cotangent. We have

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing both the numerator and denominator by $\cos A \cos B$, we obtain

$$\tan(A + B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Similarly, one can deduce that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

EXAMPLE 3.3.1. For every $x \in \mathbb{R}$, we have

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x = \cos x,$$

and

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x.$$

These form part (i) of the Properties of trigonometric functions.

Of particular interest is the special case when $A = B$.

DOUBLE ANGLE IDENTITIES. For every $x \in \mathbb{R}$, we have

(a) $\sin 2x = 2 \sin x \cos x$; and

(b) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$.

HALF ANGLE IDENTITIES. For every $y \in \mathbb{R}$, we have

$$\sin^2\left(\frac{y}{2}\right) = \frac{1 - \cos y}{2} \quad \text{and} \quad \cos^2\left(\frac{y}{2}\right) = \frac{1 + \cos y}{2}.$$

PROOF. Let $x = y/2$. Then part (b) of the Double angle identities give

$$\cos y = 1 - 2 \sin^2\left(\frac{y}{2}\right) = 2 \cos^2\left(\frac{y}{2}\right) - 1.$$

The results follow easily. ♣

EXAMPLE 3.3.2. Suppose that we wish to find the precise values of $\cos(-3\pi/8)$ and $\sin(-3\pi/8)$. We have

$$\cos^2\left(-\frac{3\pi}{8}\right) = \frac{1 + \cos(-3\pi/4)}{2}.$$

It is not difficult to show that $\cos(-3\pi/4) = -1/\sqrt{2}$, so that

$$\cos^2\left(-\frac{3\pi}{8}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2} - 1}{2\sqrt{2}}.$$

It is easy to see that $\cos(-3\pi/8) > 0$, and so

$$\cos\left(-\frac{3\pi}{8}\right) = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}.$$

Similarly, we have

$$\sin^2\left(-\frac{3\pi}{8}\right) = \frac{1 - \cos(-3\pi/4)}{2} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2} + 1}{2\sqrt{2}}.$$

It is easy to see that $\sin(-3\pi/8) < 0$, and so

$$\sin\left(-\frac{3\pi}{8}\right) = -\sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}.$$

EXAMPLE 3.3.3. Suppose that α is an angle in the first quadrant and β is an angle in the third quadrant. Suppose further that $\sin \alpha = 3/5$ and $\cos \beta = -5/13$. Using the first of the Pythagorean identities, we have

$$\cos^2 \alpha = 1 - \sin^2 \alpha = \frac{16}{25} \quad \text{and} \quad \sin^2 \beta = 1 - \cos^2 \beta = \frac{144}{169}.$$

On the other hand, using the CAST rule, we have $\cos \alpha > 0$ and $\sin \beta < 0$. It follows that $\cos \alpha = 4/5$ and $\sin \beta = -12/13$. Then

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta = \frac{3}{5} \times \left(-\frac{5}{13}\right) - \frac{4}{5} \times \left(-\frac{12}{13}\right) = \frac{33}{65}$$

and

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{4}{5} \times \left(-\frac{5}{13}\right) + \frac{3}{5} \times \left(-\frac{12}{13}\right) = -\frac{56}{65},$$

so that

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = -\frac{33}{56}.$$

EXAMPLE 3.3.4. For every $x \in \mathbb{R}$, we have

$$\sin 6x \cos 2x - \cos 6x \sin 2x = \sin(6x - 2x) = \sin 4x = 2 \sin 2x \cos 2x.$$

Note that the first step uses a difference identity, while the last step uses a double angle identity.

EXAMPLE 3.3.5. For appropriate values of $\alpha, \beta \in \mathbb{R}$, we have

$$\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta - \cos \alpha \sin \beta} = \frac{1 + \cot \alpha \tan \beta}{1 - \cot \alpha \tan \beta}.$$

Note that the first step uses sum and difference identities, while the second step involves dividing both the numerator and the denominator by $\sin \alpha \cos \beta$.

EXAMPLE 3.3.6. For every $x \in \mathbb{R}$ for which $\sin 4x \neq 0$, we have

$$\frac{\cos 8x}{\sin^2 4x} = \frac{\cos^2 4x - \sin^2 4x}{\sin^2 4x} = \cot^2 4x - 1.$$

Note that the first step involves a double angle identity.

EXAMPLE 3.3.7. For every $x \in \mathbb{R}$ for which $\cos x \neq 0$, we have

$$\frac{\sin 2x}{1 - \sin^2 x} = \frac{2 \sin x \cos x}{\cos^2 x} = 2 \tan x.$$

Note that the first step involves a double angle identity as well as a Pythagorean identity.

EXAMPLE 3.3.8. For every $x \in \mathbb{R}$ for which $\sin 3x \neq 0$ and $\cos 3x \neq 0$, we have

$$\frac{\cos^2 3x - \sin^2 3x}{2 \sin 3x \cos 3x} = \frac{\cos 6x}{\sin 6x} = \cot 6x.$$

Note that the first step involves double angle identities.

EXAMPLE 3.3.9. This example is useful in calculus for finding the derivatives of the sine and cosine functions. For every $x, h \in \mathbb{R}$ with $h \neq 0$, we have

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = (\cos x) \times \frac{\sin h}{h} + (\sin x) \times \frac{\cos h - 1}{h}$$

and

$$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = -(\sin x) \times \frac{\sin h}{h} + (\cos x) \times \frac{\cos h - 1}{h}.$$

When h is very close to 0, then

$$\frac{\sin h}{h} \approx 1 \quad \text{and} \quad \frac{\cos h - 1}{h} \approx 0,$$

so that

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x \quad \text{and} \quad \frac{\cos(x+h) - \cos x}{h} \approx -\sin x.$$

This is how we show that the derivatives of $\sin x$ and $\cos x$ are respectively $\cos x$ and $-\sin x$.

PROBLEMS FOR CHAPTER 3

1. Find the precise value of each of the following quantities, showing every step of your argument:

a) $\sin \frac{4\pi}{3}$	b) $\tan \frac{4\pi}{3}$	c) $\cos \frac{3\pi}{4}$	d) $\tan \frac{3\pi}{4}$
e) $\tan \frac{65\pi}{4}$	f) $\sin \left(-\frac{47\pi}{6} \right)$	g) $\cos \frac{25\pi}{3}$	h) $\cot \left(-\frac{37\pi}{2} \right)$
i) $\sin \frac{5\pi}{3}$	j) $\tan \frac{5\pi}{3}$	k) $\cos \left(-\frac{53\pi}{6} \right)$	l) $\cot \left(-\frac{53\pi}{6} \right)$

2. Find all solutions for each of the following equations in the intervals given, showing every step of your argument:

a) $\sin x = -\frac{1}{2}, \quad 0 \leq x < 2\pi$	b) $\sin x = -\frac{1}{2}, \quad 0 \leq x < 4\pi$
c) $\sin x = -\frac{1}{2}, \quad -\pi \leq x < \pi$	d) $\sin x = -\frac{1}{2}, \quad 0 \leq x < 3\pi$
e) $\cos^2 x = \frac{3}{4}, \quad 0 \leq x < 2\pi$	f) $\cos^2 x = \frac{3}{4}, \quad -\pi \leq x < 3\pi$
g) $\tan^2 x = 3, \quad 0 \leq x < 2\pi$	h) $\tan^2 x = 3, \quad -2\pi \leq x < \pi$
i) $\cot^2 x = 3, \quad 0 \leq x < 2\pi$	j) $\sec^2 x = 2, \quad 0 \leq x < \frac{5\pi}{2}$
k) $\cos x = -\frac{1}{2}, \quad 0 \leq x < 2\pi$	l) $\cos x = -\frac{1}{2}, \quad 0 \leq x < 4\pi$
m) $\tan^2 x = 1, \quad 2\pi \leq x < 4\pi$	n) $\sin^2 x = \frac{1}{4}, \quad \frac{\pi}{2} \leq x < 2\pi$

3. Simplify each of the following expressions, showing every step of your argument:

a) $\frac{\sin x \sec x - \sin^2 x \tan x}{\sin 2x}$	b) $\sin 3x \cos x + \sin x \cos 3x$
c) $\cos 5x \cos x + \sin x \sin 5x$	d) $\sin 3x - \cos 2x \sin x - 2 \sin x \cos^2 x$
e) $\frac{\sin 4x}{(\cos^2 x - \sin^2 x) \sin x \cos x}$	f) $\cos 4x \cos 3x - 4 \sin x \sin 3x \cos x \cos 2x$
g) $2 \sin 3x \cos 3x \cos 5x - (\cos^2 3x - \sin^2 3x) \sin 5x$	

4. We know that

$$\sin \frac{\pi}{6} = \frac{1}{2}, \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{5\pi}{24} = \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{6} \right).$$

We also know that

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \text{and} \quad \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta.$$

Use these to determine the exact values of

$$\cos \frac{5\pi}{24} \quad \text{and} \quad \sin \frac{5\pi}{24}.$$

[HINT: Put your calculators away. Your answers will be square roots of expressions involving the numbers $\sqrt{2}$ and $\sqrt{3}$.]

5. Find the precise value of each of the following quantities, showing every step of your argument:

$$\text{a) } \cos \left(-\frac{\pi}{8} \right) \quad \text{b) } \sin \left(-\frac{\pi}{8} \right) \quad \text{c) } \cos \frac{7\pi}{12} \quad \text{d) } \tan \frac{7\pi}{12}$$

6. Use the sum and difference identities for the sine and cosine functions to deduce each of the following identities:

$$\begin{array}{lll} \text{a) } \sin(x + \pi) = -\sin x & \text{b) } \cos(-x) = \cos x & \text{c) } \tan(\pi - x) = -\tan x \\ \text{d) } \cot \left(\frac{\pi}{2} + x \right) = -\tan x & & \end{array}$$

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