ELEMENTARY MATHEMATICS

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Chapter 5

POLYNOMIAL EQUATIONS

5.1. Linear Equations

Consider an equation of the type

$$ax + b = 0, \tag{1}$$

where $a, b \in \mathbb{R}$ are constants and $a \neq 0$. To solve such an equation, we first subtract b from both sides of the equation to obtain

$$ax = -b, \tag{2}$$

and then divide both sides of this latter equation by a to obtain

$$x = -\frac{b}{a}.$$

Occasionally a given linear equation may be a little more complicated than (1) or (2). However, with the help of some simple algebra, one can reduce the given equation to one of type (1) or type (2).

EXAMPLE 5.1.1. Suppose that

$$\frac{4x}{x+2} = \frac{18}{5}.$$

Multiplying both sides by $5(x + 2)$, the product of the two denominators, we obtain

$$20x = 18(x + 2) = 18x + 36.$$

Subtracting $18x$ from both sides, we obtain $2x = 36$, and so $x = 36/2 = 18$.

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† This chapter was written at Macquarie University in 1999.

Example 5.1.2. Suppose that
$$\frac{x-6}{2}+\frac{3x}{4}=x+1.$$
Multiplying both sides by 4, we obtain $2(x-6)+3x=4(x+1)$. Now $2(x-6)+3x=5x-12$ and $4(x+1)=4x+4$. It follows that $5x-12=4x+4$, so that $x-16=0$, giving $x=16$.

Example 5.1.3. Suppose that
$$\frac{6x-1}{4x+3}=\frac{3x-7}{2x-5}.$$
Multiplying both sides by $(4x+3)(2x-5)$, the product of the two denominators, we obtain
$$(6x-1)(2x-5)=(3x-7)(4x+3).$$
Now $(6x-1)(2x-5)=12x^2-32x+5$ and $(3x-7)(4x+3)=12x^2-19x-21$. It follows that $12x^2-32x+5=12x^2-19x-21$, so that $-13x+26=0$, whence $x=2$.

Example 5.1.4. Suppose that
$$\frac{3x+1}{2x+3}=\frac{3x-3}{2x+5}.$$
Multiplying both sides by $(2x+3)(2x+5)$, the product of the two denominators, we obtain
$$(3x+1)(2x+5)=(3x-3)(2x+3).$$
Now $(3x+1)(2x+5)=6x^2+17x+5$ and $(3x-3)(2x+3)=6x^2+3x-9$. Check that the original equation is equivalent to $14x+14=0$, so that $x=-1$.

We next consider a pair of simultaneous linear equations in two variables, of the type
$$\begin{aligned} a_1x+b_1y&=c_1,\\ a_2x+b_2y&=c_2, \end{aligned} \tag{3}$$
where $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$. Multiplying the first equation in (3) by b_2 and multiplying the second equation in (3) by b_1, we obtain
$$\begin{aligned} a_1b_2x+b_1b_2y&=c_1b_2,\\ a_2b_1x+b_1b_2y&=c_2b_1. \end{aligned} \tag{4}$$
Subtracting the second equation in (4) from the first equation, we obtain
$$(a_1b_2x+b_1b_2y)-(a_2b_1x+b_1b_2y)=c_1b_2-c_2b_1.$$
Some simple algebra leads to
$$(a_1b_2-a_2b_1)x=c_1b_2-c_2b_1. \tag{5}$$
On the other hand, multiplying the first equation in (3) by a_2 and multiplying the second equation in (3) by a_1, we obtain
$$\begin{aligned} a_1a_2x+b_1a_2y&=c_1a_2,\\ a_1a_2x+b_2a_1y&=c_2a_1. \end{aligned} \tag{6}$$
Subtracting the second equation in (6) from the first equation, we obtain
$$(a_1a_2x+b_1a_2y)-(a_1a_2x+b_2a_1y)=c_1a_2-c_2a_1.$$
Some simple algebra leads to
$$(b_1a_2-b_2a_1)y=c_1a_2-c_2a_1. \tag{7}$$
Suppose that $a_1b_2-a_2b_1\neq 0$. Then (5) and (7) can be written in the form
$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1} \qquad \text{and} \qquad y=\frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}. \tag{8}$$

In practice, we do not need to remember these formulae. It is much easier to do the calculations by using some common sense and cutting a few corners in doing so.

We have the following geometric interpretation. Each of the two linear equations in (3) represents a line on the xy-plane. The condition $a_1b_2 - a_2b_1 \neq 0$ ensures that the two lines are not parallel, so that they intersect at precisely one point, given by (8).

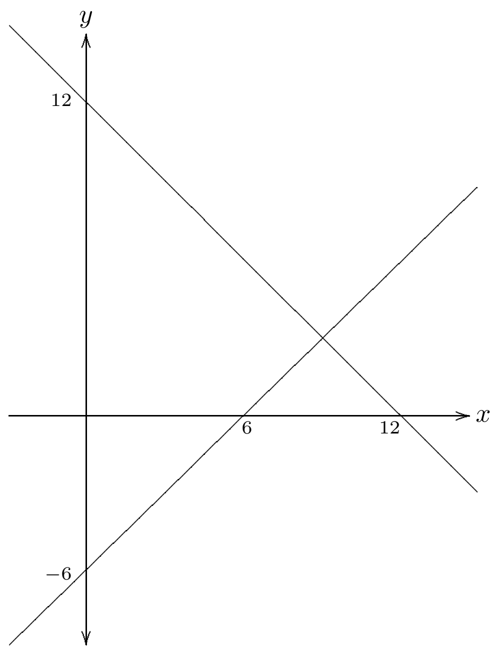
EXAMPLE 5.1.5. Suppose that

$$\begin{aligned} x + y &= 12, \\ x - y &= \ \ 6. \end{aligned}$$

Note that we can eliminate y by adding the two equations. More precisely, we have

$$(x + y) + (x - y) = 12 + 6.$$

This gives $2x = 18$, so that $x = 9$. We now substitute the information $x = 9$ into one of the two original equations. Simple algebra leads to $y = 3$. We have the following picture.



EXAMPLE 5.1.6. Suppose that

$$\begin{aligned} x + \ \ y &= 32, \\ 3x + 2y &= 70. \end{aligned}$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$\begin{aligned} 2x + 2y &= 64, \\ 3x + 2y &= 70. \end{aligned}$$

The effect of this is that both equations have a term $2y$. We now subtract the first equation from the second equation to eliminate this common term. More precisely, we have

$$(3x + 2y) - (2x + 2y) = 70 - 64.$$

This gives $x = 6$. We now substitute the information $x = 6$ into one of the two original equations. Simple algebra leads to $y = 26$.

EXAMPLE 5.1.7. Suppose that

$$\begin{aligned} 3x + 2y &= 10, \\ 4x - 3y &= 2. \end{aligned}$$

We can multiply the first equation by 4 and the second equation by 3 to obtain

$$\begin{aligned} 12x + 8y &= 40, \\ 12x - 9y &= 6. \end{aligned}$$

The effect of this is that both equations have a term $12x$. We now subtract the second equation from the first equation to eliminate this common term. More precisely, we have

$$(12x + 8y) - (12x - 9y) = 40 - 6.$$

This gives $17y = 34$, so that $y = 2$. We now substitute the information $y = 2$ into one of the two original equations. Simple algebra leads to $x = 2$. The reader may try to eliminate the variable y first and show that we must have $x = 2$.

EXAMPLE 5.1.8. Suppose that

$$\begin{aligned} 7x - 5y &= 16, \\ 2x + 7y &= 13. \end{aligned}$$

We can multiply the first equation by 7 and the second equation by 5 to obtain

$$\begin{aligned} 49x - 35y &= 112, \\ 10x + 35y &= 65. \end{aligned}$$

The effect of this is that both equations have a term $35y$ but with opposite signs. We now add the two equation to eliminate this common term. More precisely, we have

$$(49x - 35y) + (10x + 35y) = 112 + 65.$$

This gives $59x = 177$, so that $x = 3$. We now substitute the information $x = 3$ into one of the two original equations. Simple algebra leads to $y = 1$.

EXAMPLE 5.1.9. Suppose that the difference between two numbers is equal to 11, and that twice the smaller number minus 4 is equal to the larger number. To find the two numbers, let x denote the larger number and y denote the smaller number. Then we have $x - y = 11$ and $2y - 4 = x$, so that

$$\begin{aligned} x - y &= 11, \\ x - 2y &= -4. \end{aligned}$$

We now eliminate the variable x by subtracting the second equation from the first equation. More precisely, we have

$$(x - y) - (x - 2y) = 11 - (-4).$$

This gives $y = 15$. We now substitute the information $y = 15$ into one of the two original equations. Simple algebra leads to $x = 26$.

EXAMPLE 5.1.10. Suppose that a rectangle is 5cm longer than it is wide. Suppose also that if the length and width are both increased by 2cm, then the area of the rectangle increases by 50cm^2. To find the dimension of the rectangle, let x denote its length and y denote its width. Then we have $x = y+5$ and $(x+2)(y+2) - xy = 50$. Simple algebra shows that the second equation is the same as $2x+2y+4 = 50$. We therefore have

$$\begin{aligned} x - y &= 5, \\ 2x + 2y &= 46. \end{aligned}$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$\begin{aligned} 2x - 2y &= 10, \\ 2x + 2y &= 46. \end{aligned}$$

We now eliminate the variable y by adding the two equations. More precisely, we have

$$(2x - 2y) + (2x + 2y) = 10 + 46.$$

This gives $4x = 56$, so that $x = 14$. It follows that $y = 9$.

The idea of eliminating one of the variables can be extended to solve systems of three linear equations. We illustrate the ideas by the following four examples.

EXAMPLE 5.1.11. Suppose that

$$\begin{aligned} x + \ \ y + z &= \ \ 6, \\ 2x + 3y + z &= 13, \\ x + 2y - z &= \ \ 5. \end{aligned}$$

Adding the first equation and the third equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively

$$\begin{aligned} 2x + 3y &= 11, \\ 3x + 5y &= 18. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $y = 3$. Substituting back to one of the original equations, we obtain $z = 2$.

EXAMPLE 5.1.12. Suppose that

$$\begin{aligned} x - \ \ y + \ \ z &= 10, \\ 4x + 2y - 3z &= \ \ 8, \\ 3x - 5y + 2z &= 34. \end{aligned}$$

We can multiply the three equations by 6, 2 and 3 respectively to obtain

$$\begin{aligned} 6x - \ \ 6y + 6z &= \ \ 60, \\ 8x + \ \ 4y - 6z &= \ \ 16, \\ 9x - 15y + 6z &= 102. \end{aligned}$$

The reason for the multiplication is to arrange for the term $6z$ to appear in each equation to make the elimination of the variable z easier. Indeed, adding the first equation and the second equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively

$$\begin{aligned} 14x - \ \ 2y &= \ \ 76, \\ 17x - 11y &= 118. \end{aligned}$$

Multiplying the first equation by 11 and the second equation by 2, we obtain

$$\begin{aligned} 154x - 22y &= 836, \\ 34x - 22y &= 236. \end{aligned}$$

Eliminating the variable y, we obtain $120x = 600$, so that $x = 5$. It follows that $y = -3$. Using now one of the original equations, we obtain $z = 2$.

EXAMPLE 5.1.13. Suppose that

$$\begin{aligned} 6x + 4y - 2z &= 0, \\ 3x - 2y + 4z &= 3, \\ 5x - 2y + 6z &= 3. \end{aligned}$$

Multiplying the last two equations by 2, we obtain

$$\begin{aligned} 6x + 4y - \ \ 2z &= 0, \\ 6x - 4y + \ \ 8z &= 6, \\ 10x - 4y + 12z &= 6. \end{aligned}$$

The reason for the multiplication is to arrange for the term $4y$ to appear in each equation to make the elimination of the variable y easier. Indeed, adding the first equation and the second equation, or adding the first equation and the third equation, we eliminate the variable y on both occasions and obtain respectively

$$\begin{aligned} 12x + 6z &= 6, \\ 16x + 10z &= 6. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $z = -1$. Substituting back to one of the original equations, we obtain $y = -2$.

EXAMPLE 5.1.14. Suppose that

$$\begin{aligned} 2x + y - z &= 9, \\ 5x + 2z &= -3, \\ 7x - 2y &= 1. \end{aligned}$$

Our strategy here is to eliminate the variable y between the first and third equations. To do this, the first equation can be written in the form $4x + 2y - 2z = 18$. Adding this to the third equation, and also keeping the second equation as it is, we obtain

$$\begin{aligned} 11x - 2z &= 19, \\ 5x + 2z &= -3. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $z = -4$. Substituting back to one of the original equations, we obtain $y = 3$. The reader may also wish to first eliminate the variable z between the first two equations and obtain a system of two equations in x and y.

5.2. Quadratic Equations

Consider an equation of the type

$$ax^2 + bx + c = 0, \tag{9}$$

where $a, b, c \in \mathbb{R}$ are constants and $a \neq 0$. To solve such an equation, we observe first of all that

$$\begin{aligned} ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0 \end{aligned}$$

precisely when

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{10}$$

There are three cases:

(1) If $b^2 - 4ac < 0$, then the right hand side of (10) is negative. It follows that (10) is never satisfied for any real number x, so that the equation (9) has no real solution.

(2) If $b^2 - 4ac = 0$, then (10) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Indeed, this solution occurs twice, as we shall see later.

(3) If $b^2 - 4ac > 0$, then (10) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct real solutions for the equation (9).

EXAMPLE 5.2.1. For the equation $2x^2+6x+4=0$, we have $(a,b,c)=(2,6,4)$, so that $b^2-4ac=4>0$. It follows that this equation has two distinct real solutions, given by

$$x=\frac{-6\pm\sqrt{4}}{4}=-1 \text{ or } -2.$$

Observe that $2x^2+6x+4=2(x+1)(x+2)$.

EXAMPLE 5.2.2. For the equation $x^2+2x+3=0$, we have $(a,b,c)=(1,2,3)$, so that $b^2-4ac=-8<0$. It follows that this equation has no solution.

EXAMPLE 5.2.3. For the equation $3x^2-12x+12=0$, we have $b^2-4ac=0$. It follows that this equation has one real solution, given by $x=2$. Observe that $3x^2-12x+12=3(x-2)^2$. This is the reason we say that the root occurs twice.

5.3. Factorization Again

Consider equation (9) again. Sometimes we may be able to find a factorization of the form

$$ax^2+bx+c=a(x-\alpha)(x-\beta), \tag{11}$$

where $\alpha,\beta\in\mathbb{R}$. Clearly $x=\alpha$ and $x=\beta$ are solutions of the equation (9).

EXAMPLE 5.3.1. For the equation $x^2-5x=0$, we have the factorization

$$x^2-5x=x(x-5)=(x-0)(x-5).$$

It follows that the two solutions of the equation are $x=0$ and $x=5$.

EXAMPLE 5.3.2. For the equation $x^2-9=0$, we have the factorization $x^2-9=(x-3)(x+3)$. It follows that the two solutions of the equation are $x=\pm3$.

Note that

$$a(x-\alpha)(x-\beta)=a(x^2-(\alpha+\beta)x+\alpha\beta)=ax^2-a(\alpha+\beta)x+a\alpha\beta.$$

It follows from (11) that

$$ax^2+bx+c=ax^2-a(\alpha+\beta)x+a\alpha\beta.$$

Equating corresponding coefficients, we obtain

$$b=-a(\alpha+\beta) \qquad \text{and} \qquad c=a\alpha\beta.$$

We have proved the following result.

SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION. *Suppose that $x=\alpha$ and $x=\beta$ are the two roots of a quadratic equation $ax^2+bx+x=0$. Then*

$$\alpha+\beta=-\frac{b}{a} \qquad \textit{and} \qquad \alpha\beta=\frac{c}{a}.$$

EXAMPLE 5.3.3. For the equation $x^2 - 5x - 7 = 0$, we have $(a, b, c) = (1, -5, -7)$, and

$$x = \frac{5 \pm \sqrt{25 + 28}}{2} = \frac{5 \pm \sqrt{53}}{2}.$$

Note that

$$\frac{5+\sqrt{53}}{2} + \frac{5-\sqrt{53}}{2} = 5 \qquad \text{and} \qquad \frac{5+\sqrt{53}}{2} \times \frac{5-\sqrt{53}}{2} = \frac{25-53}{4} = -7.$$

EXAMPLE 5.3.4. For the equation $x^2 - 13x + 4 = 0$, we have $(a, b, c) = (1, -13, 4)$, and

$$x = \frac{13 \pm \sqrt{169 - 16}}{2} = \frac{13 \pm \sqrt{153}}{2}.$$

Note that

$$\frac{13+\sqrt{153}}{2} + \frac{13-\sqrt{153}}{2} = 13 \qquad \text{and} \qquad \frac{13+\sqrt{153}}{2} \times \frac{13-\sqrt{153}}{2} = \frac{169-153}{4} = 4.$$

We conclude this section by studying a few more examples involving factorization of quadratic polynomials. The reader may wish to study Section 1.5 again before proceeding.

EXAMPLE 5.3.5. Consider the expression $x^2 - 4x + 3$. The roots of the equation $x^2 - 4x + 3 = 0$ are given by

$$\alpha = \frac{4+\sqrt{16-12}}{2} = 3 \qquad \text{and} \qquad \beta = \frac{4-\sqrt{16-12}}{2} = 1.$$

Hence we have $x^2 - 4x + 3 = (x - 3)(x - 1)$.

EXAMPLE 5.3.6. Consider the expression $2x^2 + 5x + 2$. The roots of the equation $2x^2 + 5x + 2 = 0$ are given by

$$\alpha = \frac{-5+\sqrt{25-16}}{4} = -\frac{1}{2} \qquad \text{and} \qquad \beta = \frac{-5-\sqrt{25-16}}{4} = -2.$$

Hence we have

$$2x^2 + 5x + 2 = 2\left(x + \frac{1}{2}\right)(x + 2) = (2x + 1)(x + 2).$$

EXAMPLE 5.3.7. Consider the expression $4x^2 - x - 14$. The roots of the equation $4x^2 - x - 14 = 0$ are given by

$$\alpha = \frac{1+\sqrt{1+224}}{8} = 2 \qquad \text{and} \qquad \beta = \frac{1-\sqrt{1+224}}{8} = -\frac{7}{4}.$$

Hence we have

$$4x^2 - x - 14 = 4(x - 2)\left(x + \frac{7}{4}\right) = (x - 2)(4x + 7).$$

EXAMPLE 5.3.8. We have

$$(x + 2)^2 - (2x - 1)^2 = (x^2 + 4x + 4) - (4x^2 - 4x + 1) = x^2 + 4x + 4 - 4x^2 + 4x - 1 = -3x^2 + 8x + 3.$$

The roots of the equation $-3x^2 + 8x + 3 = 0$ are given by

$$\alpha = \frac{-8+\sqrt{64+36}}{-6} = -\frac{1}{3} \qquad \text{and} \qquad \beta = \frac{-8-\sqrt{64+36}}{-6} = 3.$$

Hence

$$(x+2)^2-(2x-1)^2=-3\left(x+\frac{1}{3}\right)(x-3)=-(3x+1)(x-3)=(3x+1)(3-x).$$

Alternatively, we can use one of the Laws on squares (writing $a=x+2$ and $b=2x-1$). We have

$$\begin{aligned}(x+2)^2-(2x-1)^2&=((x+2)-(2x-1))((x+2)+(2x-1))\\&=(x+2-2x+1)(x+2+2x-1)=(3-x)(3x+1).\end{aligned}$$

EXAMPLE 5.3.9. Consider the expression $6p-17pq+12pq^2$. Taking out a factor p, we have

$$6p-17pq+12pq^2=p(6-17q+12q^2).$$

Consider next the quadratic factor $6-17q+12q^2$. The roots of the equation $6-17q+12q^2=0$ are given by

$$\alpha=\frac{17+\sqrt{289-288}}{24}=\frac{3}{4}\qquad\text{and}\qquad\beta=\frac{17-\sqrt{289-288}}{24}=\frac{2}{3}.$$

Hence

$$6p-17pq+12pq^2=p(6-17q+12q^2)=12p\left(q-\frac{3}{4}\right)\left(q-\frac{2}{3}\right)=p(4q-3)(3q-2).$$

EXAMPLE 5.3.10. We have $10a^2b+11ab-6b=b(10a^2+11a-6)$. Consider the quadratic factor $10a^2+11a-6$. The roots of the equation $10a^2+11a-6=0$ are given by

$$\alpha=\frac{-11+\sqrt{121+240}}{20}=\frac{2}{5}\qquad\text{and}\qquad\beta=\frac{-11-\sqrt{121+240}}{20}=-\frac{3}{2}.$$

Hence

$$10a^2b+11ab-6b=b(10a^2+11a-6)=10b\left(a-\frac{2}{5}\right)\left(a+\frac{3}{2}\right)=b(5a-2)(2a+3).$$

EXAMPLE 5.3.11. Consider the expression

$$\frac{a-3}{a^2-11a+28}-\frac{a+4}{a^2-6a-7}.$$

Note that $a^2-11a+28=(a-7)(a-4)$ and $a^2-6a-7=(a-7)(a+1)$. Hence

$$\begin{aligned}\frac{a-3}{a^2-11a+28}-\frac{a+4}{a^2-6a-7}&=\frac{a-3}{(a-7)(a-4)}-\frac{a+4}{(a-7)(a+1)}\\&=\frac{(a-3)(a+1)}{(a-7)(a-4)(a+1)}-\frac{(a+4)(a-4)}{(a-7)(a-4)(a+1)}\\&=\frac{(a-3)(a+1)-(a+4)(a-4)}{(a-7)(a-4)(a+1)}=\frac{(a^2-2a-3)-(a^2-16)}{(a-7)(a-4)(a+1)}\\&=\frac{13-2a}{(a-7)(a-4)(a+1)}.\end{aligned}$$

EXAMPLE 5.3.12. We have

$$\begin{aligned}&\frac{1}{x+1}+\frac{1}{(x+1)(x+2)}-\frac{4}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+2)(x+3)}{(x+1)(x+2)(x+3)}+\frac{(x+3)}{(x+1)(x+2)(x+3)}-\frac{4}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+2)(x+3)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{(x^2+5x+6)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{x^2+6x+5}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+1)(x+5)}{(x+1)(x+2)(x+3)}=\frac{x+5}{(x+2)(x+3)}.\end{aligned}$$

5.4. Higher Order Equations

For polynomial equations of degree greater than 2, we do not have general formulae for their solutions. However, we may occasionally be able to find some solutions by inspection. These may help us find other solutions. We shall illustrate the technique here by using three examples.

EXAMPLE 5.4.1. Consider the equation $x^3 - 4x^2 + 2x + 1 = 0$. It is easy to see that $x = 1$ is a solution of this cubic polynomial equation. It follows that $x - 1$ is a factor of the polynomial $x^3 - 4x^2 + 2x + 1$. Using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2- & 3x- & 1 & \\ \cline{2-4}
x-\ 1\,) & x^3-\ 4x^2+ & 2x+ & 1 & \\
 & x^3-\ x^2 & & & \\ \cline{2-4}
 & -\ 3x^2+ & 2x & & \\
 & -\ 3x^2+ & 3x & & \\ \cline{2-4}
 & - & x+ & 1 & \\
 & - & x+ & 1 & \\ \cline{3-4}
\end{array}
$$

Hence $x^3 - 4x^2 + 2x + 1 = (x - 1)(x^2 - 3x - 1)$. The other two roots of the equation are given by the two roots of $x^2 - 3x - 1 = 0$. These are

$$x = \frac{3 \pm \sqrt{9 + 4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$

EXAMPLE 5.4.2. Consider the equation $x^3 + 2x^2 - 5x - 6 = 0$. It is easy to see that $x = -1$ is a solution of this cubic polynomial equation. It follows that $x + 1$ is a factor of the polynomial $x^3 + 2x^2 - 5x - 6$. Using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2+ & x- & 6 & \\ \cline{2-4}
x+\ 1\,) & x^3+2x^2- & 5x- & 6 & \\
 & x^3+\ x^2 & & & \\ \cline{2-4}
 & x^2- & 5x & & \\
 & x^2+ & x & & \\ \cline{2-4}
 & - & 6x- & 6 & \\
 & - & 6x- & 6 & \\ \cline{3-4}
\end{array}
$$

Hence $x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6)$. The other two roots of the equation are given by the two roots of $x^2 + x - 6 = 0$. These are

$$x = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = 2 \text{ or } -3.$$

EXAMPLE 5.4.3. Consider the equation $x^4 + 7x^3 - 6x^2 - 2x = 0$. It is easy to see that $x = 0$ and $x = 1$ are solutions of this biquadratic polynomial equation. It follows that $x(x - 1)$ is a factor of the polynomial $x^4 + 7x^3 - 6x^2 - 2x$. Clearly we have $x^4 + 7x^3 - 6x^2 - 2x = x(x^3 + 7x^2 - 6x - 2)$. On the other hand, using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2+ & 8x+ & 2 & \\ \cline{2-4}
x-\ 1\,) & x^3+7x^2- & 6x- & 2 & \\
 & x^3-\ x^2 & & & \\ \cline{2-4}
 & 8x^2- & 6x & & \\
 & 8x^2- & 8x & & \\ \cline{2-4}
 & & 2x- & 2 & \\
 & & 2x- & 2 & \\ \cline{3-4}
\end{array}
$$

Hence $x^4 + 7x^3 - 6x^2 - 2x = x(x-1)(x^2+8x+2)$. The other two roots of the equation are given by the two roots of $x^2 + 8x + 2 = 0$. These are

$$x = \frac{-8 \pm \sqrt{64-8}}{2} = \frac{-8 \pm \sqrt{56}}{2} = -4 \pm \sqrt{14}.$$

Problems for Chapter 5

1. Solve each of the following equations:
   a) $\dfrac{9x-8}{8x-6} = \dfrac{1}{2}$ b) $\dfrac{5-2x}{3x+7} = 9$ c) $\dfrac{5x+14}{3x+2} = 3$
   d) $\dfrac{2x-2}{3} + \dfrac{10-2x}{6} = 2x-4$ e) $\dfrac{3x-4}{6x-10} = \dfrac{4x+1}{8x-7}$

2. Solve each of the following systems of linear equations:
   a) $3x - 8y = 1$, $2x + 3y = 9$ b) $4x - 3y = 14$, $9x - 4y = 26$ c) $6x - 2y = 14$, $2x + 3y = 12$ d) $5x + 2y = 4$, $7x + 3y = 5$

3. A rectangle is 2 metres longer than it is wide. On the other hand, if each side of the rectangle is increased by 2 metres, then the area increases by 16 square metres. Find the dimension of the rectangle.

4. A rectangle is 10 metres wider than it is long. On the other hand, if the width and length are both decreased by 5 metres, then the area of the rectangle decreases by 125 square metres. Find the dimension of the rectangle.

5. The lengths of the two perpendicular sides of a right-angled triangle differ by 6 centimetres. On the other hand, if the length of the longer of these two sides is increased by 3 centimetres and the length of the shorter of these two sides is decreased by 2 centimetres, then the area of the right-angled triangle formed is decreased by 5 square centimetres. What is the dimension of the original triangle?

6. Solve each of the following systems of linear equations:
   a) $3x - 2y + 4z = 11$, $2x + 3y + 3z = 17$, $4x + 6y - 2z = 10$
   b) $5x + 2y + 4z = 35$, $2x - 3y + 2z = 19$, $3x + 5y + 3z = 19$
   c) $3x + 4y + 2z = 9$, $5x - 2y + 4z = 7$, $2x + 6y - 2z = 6$
   d) $x + 4y - 3z = 2$, $x - 2y + 2z = 1$, $x + 6y - 5z = 2$

7. Determine the number of solutions of each of the following quadratic equations and find the solutions:
   a) $3x^2 - x + 1 = 0$ b) $4x^2 + 12x + 9 = 0$ c) $18x^2 - 84x + 98 = 0$
   d) $6x^2 - 13x + 6 = 0$ e) $5x^2 + 2x + 1 = 0$ f) $x^2 - 2x - 48 = 0$
   g) $12x^2 + 12x + 3 = 0$ h) $2x^2 - 32x + 126 = 0$ i) $3x^2 + 6x + 15 = 0$
   j) $16x^2 + 8x - 3 = 0$ k) $x^2 + 2x + 2 = 0$

8. Factorize each of the following expressions:
   a) $14x^2 + 19x - 3$ b) $6x^2 + x - 12$ c) $(5x+1)^2 - 20x$
   d) $(2x+1)^2 + x(2+4x)$ e) $4x^3 + 9x^2 + 2x$ f) $x^2y - xy - 6y$
   g) $8x - 2xy - xy^2$

9. Simplify each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{2}{x-2} + \dfrac{2}{x-5} - \dfrac{x}{x^2-3x+2} - \dfrac{2}{x-1}$ b) $\dfrac{x-3}{x^2-3x+2} - \dfrac{x-2}{x^2-4x+3}$
   c) $\dfrac{x^4 + x^2y^2 + y^4}{x^3 - y^3} \div \dfrac{x^2 - xy + y^2}{x-y}$ d) $\dfrac{4xy}{(x-y)^2} + \dfrac{x^2 - xy}{x^2 - y^2}\left(1 + \dfrac{y}{x}\right)$

10. Study each of the following equations for real solutions:
a) $x^3 - 6x^2 + 11x - 6 = 0$
b) $x^3 - 3x^2 + 4 = 0$
c) $x^3 + 2x^2 + 6x + 5 = 0$
d) $x^3 - x^2 - x + 1 = 0$
e) $x^3 + 2x^2 - x - 2 = 0$

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